A Call-By-Push-Value FPC and its interpretation in Linear Logic

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Abstract

We present and study a functional calculus similar to Levy's Call-By-Push-Value lambda-calculus, extended with fix-points and recursive types. We explain its connection with Linear Logic by presenting a denotational interpretation of the language in any model of Linear Logic equipped with a notion of embedding retraction pairs. We consider the particular case of the Scott model of Linear Logic from which we derive an intersection type system for our CBPV FPC and prove an adequacy theorem. Last, we introduce a fully polarized version of CBPV which is closer to Levy's original calculus, turns out to be a term language for a large fragment of Laurent's LLP and refines Parigot's lambda-mu.

Keywords lambda-calculus, call by push value, linear logic, denotational semantics, Scott semantics

Introduction

Linear Logic (LL) has been introduced as a refinement of Intuitionistic Logic: in [12], Girard proposed a very simple and natural translation of intuitionistic logic and of the λ-calculus in LL. From a categorical point of view, as explained in [24], this translation corresponds to the construction of the Kleisli category of the exponential comonad "!" of LL. An adequate categorical axiomatization of the denotational models of LL has been then provided in [2], see also [22] for a very complete and detailed picture.

In [12], another possible translation of intuitionistic formulas strangely called "boring" is mentioned. It appeared later that, just as the original Girard's translation corresponds to the call-by-name (CBN) evaluation strategy of the λ-calculus, the "boring" translation corresponds to the call-by-value (CBV) reduction strategy, see in particular [21]. Indeed, a first observation is that this latter translation does not preserve all β-reductions, but only those respecting a CBV discipline. More deeply, domain-theoretic denotational models of the λ-calculus arising through the original Girard translation, that is, arising as Kleisli categories of the exponential comonad, enjoy an adequacy property expressing that a term reduces to a "value" (a head-normal term, say) iff its interpretation is different from $\perp$. A similar property holds for the CBV translation (now, a closed value is an abstraction) with respect to the CBV reduction strategy.

Both translations give a particularly proeminent role to the Kleisli category of the "!" comonad. This is obvious for the original CBN translation but it is also true for the CBV translation if we consider that the "!" functor defines a strong monad on the cartesian closed Kleisli category: then the CBV translation coincides with Moggi's interpretation of the CBV λ-calculus in a CCC equipped with a computational monad [23].

So LL (more precisely, MELL, that is Multiplicative Exponential Linear Logic) provides a common setting where both CBN and CBV can be faithfully interpreted. In spite of its appealing symmetries and its high degree of asynchrony, the syntax of MELL proof nets is complex and does not seem to be a convenient starting point for the design of programming languages; it is rather a powerful tool for analyzing the operational and denotational properties of programming languages. It seems therefore natural to look for λ-calculi admitting a translation in MELL and where both CBN and CBV can be embedded, factorizing the two translations mentioned above. It turns out that Levy introduced a few years ago a λ-calculus subsuming both CBN and CBV: the *Call-By-Push-Value* λ-calculus (CBPV) of [18, 20], that we will show to provide a suitable such factorization.

We will recast a version of the CBPV λ-calculus within LL, or more precisely, within a polarized extension of LL. The LL exponential "!" allows to turn a term of type A into a term of type $!A$ which is duplicable and discardable, by means of an operation called *promotion*. This discardability and duplicability is made possible by the structural rules $!A$ is equipped with. It was already observed by Girard in [13] (and probably much earlier) that the property of "being equipped with structural rules" is preserved by the $\otimes$ and $\oplus$ connectives of LL. These observations can be made more accurate as follows: a "type equipped with structural rules" is a coalgebra for the "!" connective[1], that is, an object of the Eilenberg-Moore category of "!", and this category admits $\oplus$ as coproduct and $\otimes$ as cartesian product. This category is not a CCC in general but contains the CCC Kleisli category of "!" as a full subcategory (remember that the Kleisli category is the category of free !-coalgebras). This category was used crucially by Girard to give a semantics to a classical sequent calculus and by other authors (see for instance [17]) to interpret classical extensions of the λ-calculus such as Parigot's $\lambda\mu$-calculus.

[1] Girard considered actually $\otimes$-commutative comonoids, a notion which has the good taste of being independent of any choice of "!" modality. This is however not sufficient for translating classical logic in LL, unless one restricts ones attention to the free exponential modality as he did, in the framework of coherence spaces.

Replacing the Kleisli category with the larger Eilenberg-Moore category of "!" when interpreting the λ-calculus has other major benefits. Consider for instance the interpretation of ordinary PCF in an LL-induced categorical model, that is, in a Kleisli category of the "!" comonad of a categorical model $\mathcal{L}$ of LL. The simplest and most natural interpretation for the type of natural numbers is $\mathsf{N} = 1 \oplus 1 \oplus \cdots$ (ω copies of the unit 1 of the tensor product). But 1 has a canonical structure of !-coalgebra (because $1 = !\top$ where $\top$ is the terminal object of $\mathcal{L}$) and this is therefore also true of N, as a coproduct of coalgebras. This means that we have a well behaved morphism $\mathrm{h}_\mathsf{N} \in \mathcal{L}(\mathsf{N}, !\mathsf{N})$ which allows to turn any morphism $f \in \mathcal{L}_!(\mathsf{N}, X)$ of the Kleisli CCC $\mathcal{L}_!$ where we interpret PCF into a *linear* morphism $f\,\mathrm{h}_\mathsf{N} \in \mathcal{L}(\mathsf{N}, X)$. Operationally, this means that, in spite of the fact that PCF is a CBN language and that its interpretation in $\mathcal{L}_!$ is a CBN interpretation, we can deal with the terms of ground type in a CBV fashion. For instance we can replace the ordinary PCF "if zero" conditional with the following more sensible one:

$$\frac{\mathcal{P} \vdash M : \iota \qquad \mathcal{P} \vdash N : \sigma \qquad \mathcal{P}, x : \iota \vdash N' : \sigma}{\mathcal{P} \vdash \mathsf{if}(M, N, (x)N') : \sigma}$$

with the reduction rules

$$\mathsf{if}(\underline{0}, N, (x)N') \to N \qquad \mathsf{if}(\underline{n+1}, N, (x)N') \to N'\,[\underline{n}/x]$$
$$M \to M' \Rightarrow \mathsf{if}(M, N, (x)N') \to \mathsf{if}(M', N, (x)N')$$

The denotational interpretation of $\mathsf{if}(M, N, (x)N')$ uses crucially the coalgebra structure h_N of N. This idea is also reminiscent of *storage operators* of [16] which have exactly the same purpose of allowing a CBV discipline for data types in a globally CBN calculus.

We see CBPV as a very nice generalization of this idea. We consider two classes of types: the *positive types* $\varphi, \psi \ldots$ and the larger class of *general types* $\sigma, \tau, \ldots$. Just as in [13], positive types correspond to objects of $\mathcal{L}^!$ whereas general types are just objects of $\mathcal{L}$. Of course there is an obvious way of considering a positive type as a general one by simply forgetting its coalgebra structure. There is also a way of turning a general type into a positive one, using the "!" comonad, and positive types are stable under sums $\oplus$ and product $\otimes$. In Girard's CBN translation, $\sigma \Rightarrow \tau$ becomes $!\sigma \multimap \tau$: *the main idea of CBPV is to generalize this idea by allowing to replace the subtype $!\sigma$ with an arbitrary positive type φ.* Therefore, the implication of the CBPV λ-calculus is linear: all the required non-linearity is provided by the positivity of the premise. Accordingly all variables have positive types.

There is however a subtlety which does not occur in the CBN situation: consider a term M of type σ with one free variable x of positive type φ. Consider also a closed term N of type φ. The interpretation of M is a morphism $f \in \mathcal{L}(\varphi, \sigma)$ (identifying types with their interpretation) and the interpretation of N is a morphism $g \in \mathcal{L}(1, \varphi)$. There is no reason however for g to belong to $\mathcal{L}^!(1, \varphi)$ and so we cannot be sure that $f\,g$ will coincide with the interpretation of $M\,[N/x]$. Indeed, for that substitutivity property to hold, we would need g to be duplicable and discardable which is the case if we can make sure that $g \in \mathcal{L}^!(1, \varphi)$, but does not hold in general. It is here that the syntactic notion of *value* comes in: if $\varphi = !\sigma$ then N is a value if $N = R^!$ (a promotion of a term R of type σ), if $\varphi = \varphi_1 \otimes \varphi_2$ then N is a value if $N = \langle V_1, V_2 \rangle$ where V_i is a value of type φ_i and similarly for sums. The main property of values is that their interpretations are coalgebra morphisms: if N is a value then $g \in \mathcal{L}^!(1, \varphi)$. This is why, in the CBPV λ-calculus, β reduction is restricted to the case where the argument is a value: we have $\langle \lambda x^\varphi\, M \rangle V \to M\,[V/x]$ only when V is a value because we are sure in that case that the interpretation of V is a coalgebra morphism. This also means that in $\langle \lambda x^\varphi\, M \rangle N$ we need to reduce N to a value before reducing the β-redex. If, for instance, the reduction of N diverges without reaching a value then the reduction of $\langle \lambda x^\varphi\, M \rangle N$ diverges even if M does not use x, just as in CBV.

Positive types are also stable under fix-points if we assume that the objects of $\mathcal{L}$ can be equipped with a notion of embedding-retraction pairs, which is usually the case in categories of domains. Under the assumption that this new category has all countable directed colimits and that the functors interpreting types are continuous, it is easy to prove that all "positive types with parameters" have fix-points which are themselves positive types. The presence of "!" as an explicit type constructor in CBPV allows to define lazy recursive types such as $\rho = \varphi \otimes !\rho$ where φ is a given type: ρ is the type of streams of elements of type φ. At the level of terms, the construction which introduces an "!" is the aforementioned construction $R^!$ which corresponds to the well known (generalized) promotion of LL aka *exponential box*; this construction corresponds here to scheme's *thunks* or *suspensions*. In this stream type ρ the box construction is crucially used to postpone the evaluation of the tail of the stream. The box can be opened by means of an explicit *dereliction* syntactic construct $\mathsf{der}(M)$ which can be applied to any term M of type $!\sigma$ for some σ and which corresponds exactly to the usual dereliction rule of LL.

Contents. We define a simply typed CBPV calculus featuring positive and ordinary types, recursive positive types and with a fix-point operator for terms. Since many data types can be defined easily in this language (ordinary integers, lazy integers, lists, streams, various kinds of finite and infinite trees. . .), it widely encompasses PCF and is closer to a language such as FPC of [11], with the additional feature that it allows to freely combine CBV and CBN. We define the syntax of the language, provide a typing system and a simple operational semantics which is "weak" in the sense that reduction is forbidden under λs and within boxes (more general reductions can of course be defined but, just as in CBV, a let construction or new reduction rules as in [3] should be added).

Then we recall the general definition of a categorical model of LL (with fix-point operators), of its Eilenberg Moore category and we describe the additional categorical structure which will allow to interpret recursive positive types. In order to illustrate the connection of CBPV with LL without introducing proof-nets (this would be very interesting but would require more space), we describe the interpretation of our CBPV FPC in such a categorical model of LL and state a Soundness Theorem. The exact connection of this semantics with the adjunction semantics of [19] has still to be explored. Even if, eventually, the outcome of this study will be that our LL interpretation arises as a special case of Levy's we believe that it is worth being further studied because LL admits many interesting extensions (Ludics, Differential LL, Light LL etc) on which we think that CBPV will shed a new light.

We consider then the particular case where $\mathcal{L}$ is a category of prime-algebraic complete lattices and linear functions, a well known model of LL whose Kleisli category is a CCC of Scott continuous functions. We provide a very simple description of the Eilenberg-Moore category as a category of algebraic predomains and Scott continuous functions and prove an Adequacy Theorem. We also provide a description of this interpretation as a very simple "intersection" typing system. This adequacy result shows that the weak reduction is complete in the sense that if a closed term is denotationally equal to a value, then it reduces to a value for the weak reduction. We conclude by introducing a polarized version of CBPV, a calculus which is closer to original Levy's CBPV (because in this calculus all terms of positive types are "data" and are therefore freely discardable and duplicable) but generalizes it by allowing "classical" constructions borrowed from Parigot's $\lambda\mu$-

calculus. Last we define an encoding of CbPV into $\mu\mathsf{CbPV}$ and outline its basic features.

Our initial motivation for introducing $\mu\mathsf{CbPV}$ was to combine our representation of data as !-coalgebra morphisms in CbPV with the representation of stacks (continuations) as !-coalgebra morphisms in the semantics of classical calculi such as the CBN $\lambda\mu$-calculus. The $\mu\mathsf{CbPV}$ calculus we arrived to is presented in a $\lambda\mu\tilde{\mu}$ style introduced in [4] and further developed in [5]. We think that it provides a satisfactory answer to our quest and deserves further studies. Independently, Pierre-Louis Curien introduced recently in the unpublished note [6] a similar formalism for representing Levy's original CBPV (Curien's calculus however is intuitionistic whereas ours is classical).

1. Syntax

Our choices of notations are different from Levy's because we want to insist from the beginning on the similarity with basic LL constructs.

Types are given by the following BNF syntax. We define by mutual induction two kinds of types: *positive types* (denoted with letters $\varphi, \psi \dots$) and *general types* (denoted with letters $\sigma, \tau \dots$). We assume to be given type variables $\zeta, \xi \dots$.

$$\varphi, \psi, \dots := !\sigma \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \mathsf{Fix}\,\zeta \cdot \varphi \qquad (1)$$

$$\sigma, \tau \dots := \varphi \mid \varphi \multimap \sigma \mid \top \qquad (2)$$

We consider the types up to the equation $\mathsf{Fix}\,\zeta \cdot \varphi = \varphi\,[\mathsf{Fix}\,\zeta \cdot \varphi/\zeta]$. One could also consider more general recursive types allowing the construction $\mathsf{Fix}\,\zeta \cdot \sigma$ for σ a general type. In this paper we restrict to positive recursive types.

Terms are given by the following BNF syntax, assuming to be given variables $x, y, \dots$.

$$\begin{aligned} M, N \dots := x \mid M^! \mid \langle M, N\rangle \mid \mathsf{in}_1 M \mid \mathsf{in}_2 M \\ \mid \lambda x^{\varphi}\, M \mid \langle M\rangle N \mid \mathsf{case}(M, (x_1)N_1, (x_2)N_2) \\ \mid \mathsf{pr}_1 M \mid \mathsf{pr}_2 M \mid \mathsf{der}(M) \mid \mathsf{fix}\, x^{!\sigma}\, M \end{aligned}$$

The notion of substitution is defined as usual. We provide now typing rules for these terms. A typing context is an expression $\mathcal{P} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$ where all types are positive and the x_is are pairwise distinct variables. The typing rules are given in Figure 1.

Remark: It might seem strange to the reader acquainted with LL that the rules introducing the $\otimes$ connective and eliminating the $\multimap$ connective have an "additive" handling of typing contexts (by this we mean that the same typing context $\mathcal{P}$ occurs in both premises). The reason for this will become clear in Section 2 where we shall see that positive types are interpreted as !-coalgebras which are equipped with morphisms allowing to interpret the structural rules of weakening and contraction. Remember that typing contexts involve positive types only.

We define now a *weak* reduction relation on terms, meaning that we never reduce within a "box" $M^!$ or under a λ. Dealing with more general reductions will require to extend the syntax with explicit substitutions or with a let constructions, or to add commutation reduction rules in the spirit of σ-equivalence; this will be done in further work.

Before giving the reduction rules, we have to define the notion of *value* as follows:

- if x is a variable then x is a value
- for any term M, the term $M^!$ is a value
- if M is a value then $\mathsf{in}_i M$ is a value for $i = 1, 2$
- if M_1 and M_2 are values then $\langle M_1, M_2\rangle$ is a value.

Remark: A closed value is simply a tree whose leaves are "boxes" or "thunks" $M^!$ (where the M's are arbitrary well typed closed terms) and whose internal nodes are either unary nodes bearing an index 1 or 2, and ordered binary nodes.

We use letters $V, W \dots$ to denote values. The reduction relation is defined Figure 2.

Proposition 1 *If V is a value, there is no term M such that $V \to_{\mathsf{w}} M$.*

Proof. Straightforward induction on V. □

Proposition 2 *The reduction relation $\to_{\mathsf{w}}$ enjoys subject reduction and Church-Rosser.*

The first statement is a straightforward verification using a Substitution Lemma that we do not state. The second one is easy because $\to_{\mathsf{w}}$ has actually the diamond property.

1.1 Examples

Given any type σ, we define $\Omega^\sigma = \mathsf{fix}\, x^{!\sigma}\, \mathsf{der}(x)$ which satisfies $\vdash \Omega^\sigma : \sigma$. It is clear that $\Omega^\sigma \to_{\mathsf{w}} \mathsf{der}((\Omega^\sigma)^!) \to_{\mathsf{w}} \Omega^\sigma$ so that we can consider Ω^σ as the ever-looping program of type σ.

Unit type and natural numbers. We define a unit type 1 by $1 = !\top$, and we set $* = (\Omega^\top)^!$. We define the type ι of unary natural numbers by $\iota = 1 \oplus \iota$ (by this we mean that $\iota = \mathsf{Fix}\,\zeta \cdot (1 \oplus \zeta)$). We define $\underline{0} = \mathsf{in}_1 *$ and $\underline{n+1} = \mathsf{in}_2 \underline{n}$ so that we have $\mathcal{P} \vdash \underline{n} : \iota$ for each $n \in \mathbb{N}$.

Then, given a term M, we define the term $\mathsf{suc}(M) = \mathsf{in}_2 M$, so that we have

$$\frac{\mathcal{P} \vdash M : \iota}{\mathcal{P} \vdash \mathsf{suc}(M) : \iota}$$

Last, given terms M, N_1 and N_2 and a variable x, we define an "ifz" conditional by $\mathsf{if}(M, N_1, (x)N_2) = \mathsf{case}(M, (z)N_1, (x)N_2)$ where z is not free in N_1, so that

$$\frac{\mathcal{P} \vdash M : \iota \qquad \mathcal{P} \vdash N_1 : \sigma \qquad \mathcal{P}, x : \iota \vdash N_2 : \sigma}{\mathcal{P} \vdash \mathsf{if}(M, N_1, (x)N_2) : \sigma}$$

Streams. Let φ be a positive type and S_φ be the positive type defined by $\mathsf{S}_\varphi = \varphi \otimes !\mathsf{S}_\varphi$, that is $\mathsf{S}_\varphi = \mathsf{Fix}\,\zeta \cdot (\varphi \otimes !\zeta)$. We can define a term M such that $\vdash M : \mathsf{S}_\varphi \multimap \iota \multimap \varphi$ which computes the nth element of a stream:

$$\begin{aligned} M = \mathsf{fix}\, f^{!(\mathsf{S}_\varphi \multimap \iota \multimap \varphi)}\, \lambda x^{\mathsf{S}_\varphi}\, \lambda y^{\iota} \\ \mathsf{if}(y, \mathsf{pr}_1 x, (z)\langle \mathsf{der}(f)\rangle \mathsf{der}(\mathsf{pr}_2 x)\, z) \end{aligned}$$

Conversely, we can define a term N such that $\vdash N : !(\iota \multimap \varphi) \multimap \mathsf{S}_\varphi$ which turns a function into a stream.

$$\begin{aligned} N = \mathsf{fix}\, F^{!(!(\iota \multimap \varphi) \multimap \mathsf{S}_\varphi)}\, \lambda f^{!(\iota \multimap \varphi)} \\ \langle \langle \mathsf{der}(f)\rangle \underline{0}, (\langle \mathsf{der}(F)\rangle (\lambda x^{\iota}\, \langle \mathsf{der}(f)\rangle \mathsf{suc}(x))^!)^! \rangle \end{aligned}$$

Observe that the recursive call of F is encapsulated into a box, which makes the construction lazy.

Lists. There are various possibilities for defining a type of lists of elements of a positive type φ. The simplest definition is $\lambda_0 = 1 \oplus (\varphi \otimes \lambda_0)$. This corresponds to the ordinary ML type of lists. But we can also define $\lambda_1 = 1 \oplus (\varphi \otimes !\lambda_1)$ and then we have a type of lazy lists where the tail of the list is computed only when required (this type contains also streams).

We could also consider $\lambda_2 = 1 \oplus (!\sigma \otimes \lambda_2)$ which allows to manipulate lists of objects of type σ (which can be a general type) without accessing their elements.

$$\frac{\mathcal{P} \vdash M : \sigma}{\mathcal{P} \vdash M^! : !\sigma} \quad \frac{\mathcal{P} \vdash M_1 : \varphi_1 \qquad \mathcal{P} \vdash M_2 : \varphi_2}{\mathcal{P} \vdash \langle M_1, M_2 \rangle : \varphi_1 \otimes \varphi_2} \quad \frac{\mathcal{P} \vdash M : \varphi_i}{\mathcal{P} \vdash \mathsf{in}_i M : \varphi_1 \oplus \varphi_2} \quad \frac{}{\mathcal{P}, x : \varphi \vdash x : \varphi}$$

$$\frac{\mathcal{P}, x : \varphi \vdash M : \sigma}{\mathcal{P} \vdash \lambda x^\varphi\, M : \varphi \multimap \sigma} \quad \frac{\mathcal{P} \vdash M : \varphi \multimap \sigma \qquad \mathcal{P} \vdash N : \varphi}{\mathcal{P} \vdash \langle M \rangle N : \sigma} \quad \frac{\mathcal{P} \vdash M : !\sigma}{\mathcal{P} \vdash \mathsf{der}(M) : \sigma} \quad \frac{\mathcal{P}, x : !\sigma \vdash M : \sigma}{\mathcal{P} \vdash \mathsf{fix}\, x^{!\sigma}\, M : \sigma}$$

$$\frac{\mathcal{P} \vdash M : \varphi_1 \oplus \varphi_2 \qquad \mathcal{P}, x_1 : \varphi_1 \vdash M_1 : \sigma \qquad \mathcal{P}, x_2 : \varphi_2 \vdash M_2 : \sigma}{\mathcal{P} \vdash \mathsf{case}(M, (x_1) M_1, (x_2) M_2) : \sigma} \quad \frac{\mathcal{P} \vdash M : \varphi_1 \otimes \varphi_2}{\mathcal{P} \vdash \mathsf{pr}_i M : \varphi_i}$$

Figure 1. Typing system for CbPV

$$\frac{}{\mathsf{der}(M^!) \to_{\mathsf{w}} M} \quad \frac{}{\langle \lambda x^\varphi\, M \rangle V \to_{\mathsf{w}} M\,[V/x]} \quad \frac{}{\mathsf{pr}_i \langle V_1, V_2 \rangle \to_{\mathsf{w}} V_i} \quad \frac{}{\mathsf{case}(\mathsf{in}_i V, (x_1) M_1, (x_2) M_2) \to_{\mathsf{w}} M_i\,[V/x_i]}$$

$$\frac{}{\mathsf{fix}\, x^{!\sigma}\, M \to_{\mathsf{w}} M\left[(\mathsf{fix}\, x^{!\sigma}\, M)^!/x\right]} \quad \frac{M \to_{\mathsf{w}} M'}{\mathsf{der}(M) \to_{\mathsf{w}} \mathsf{der}(M')} \quad \frac{M \to_{\mathsf{w}} M'}{\langle M \rangle N \to_{\mathsf{w}} \langle M' \rangle N} \quad \frac{N \to_{\mathsf{w}} N'}{\langle M \rangle N \to_{\mathsf{w}} \langle M \rangle N'}$$

$$\frac{M \to_{\mathsf{w}} M'}{\mathsf{pr}_i M \to_{\mathsf{w}} \mathsf{pr}_i M'} \quad \frac{M_1 \to_{\mathsf{w}} M_1'}{\langle M_1, M_2 \rangle \to_{\mathsf{w}} \langle M_1', M_2 \rangle} \quad \frac{M_2 \to_{\mathsf{w}} M_2'}{\langle M_1, M_2 \rangle \to_{\mathsf{w}} \langle M_1, M_2' \rangle} \quad \frac{M \to_{\mathsf{w}} M'}{\mathsf{in}_i M \to_{\mathsf{w}} \mathsf{in}_i M'}$$

$$\frac{M \to_{\mathsf{w}} M'}{\mathsf{case}(M, (x_1) M_1, (x_2) M_2) \to_{\mathsf{w}} \mathsf{case}(M', (x_1) M_1, (x_2) M_2)}$$

Figure 2. Weak reduction axioms and rules for CbPV

2. Denotational Semantics

The kind of denotational models we are interested in in this paper are those induced by a model of LL, in the spirit of Girard's seminal work [13] on the semantics of the classical system LC where positive formulas are interpreted as $\otimes$-comonoids; this interpretation is further developed *eg.* in [17]. We use here exactly the same idea for interpreting positive types.

We first recall the general categorical definition of a model of LL implicit in [12], our main reference here is [22] to which we also refer for the rich bibliography on this general topic.

2.1 Models of Linear Logic

A model of LL consists of the following data.

- A category $\mathcal{L}$.
- A symmetric monoidal structure $(\otimes, 1, \lambda, \rho, \alpha, \sigma)$ which is assumed to be closed: $\otimes$ is a functor $\mathcal{L}^2 \to \mathcal{L}$, 1 an object of $\mathcal{L}$, $\lambda_X \in \mathcal{L}(1 \otimes X, X)$, $\rho_X \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X,Y,Z} \in \mathcal{L}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ and $\sigma_{X,Y} \in \mathcal{L}(X \otimes Y, Y \otimes X)$ are natural isomorphisms satisfying coherence diagrams that we do not record here. We use $X \multimap Y$ for the object of linear morphisms from X to Y, ev for the evaluation morphism which belongs to $\mathcal{L}((X \multimap Y) \otimes X, Y)$ and cur for the linear curryfication map $\mathcal{L}(Z \otimes X, Y) \to \mathcal{L}(Z, X \multimap Y)$.
- An object $\perp$ of $\mathcal{L}$ such that the natural morphism $\eta_X = \mathrm{cur}(\mathrm{ev}\ \sigma_{X \multimap \perp, X}) \in \mathcal{L}(X, (X \multimap \perp) \multimap \perp)$ is an iso for each object X (one says that $\mathcal{L}$ is a $*$-autonomous category). We use $X^\perp$ for the object $X \multimap \perp$ of $\mathcal{L}$.
- The category $\mathcal{L}$ is assumed to be cartesian. We use $\top$ for the terminal object, $\&$ for the cartesian product and pr_i for the projections. It follows by $*$-autonomy that $\mathcal{L}$ has also all finite coproducts. We use 0 for the initial object, $\oplus$ for the coproduct and in_i for the injections. Given an object X of $\mathcal{L}$, we use in^X for the unique element of $\mathcal{L}(0, X)$.
- We are also given a comonad $!\_ : \mathcal{L} \to \mathcal{L}$ with counit $\mathrm{der}_X \in \mathcal{L}(!X, X)$ (called dereliction) and comultiplication $\mathrm{dig}_X \in \mathcal{L}(!X, !!X)$ (called digging).
- And a strong symmetric monoidal structure for the functor $!\_$, from the symmetric monoidal category $(\mathcal{L}, \&)$ to the symmetric monoidal category $(\mathcal{L}, \otimes)$. This means that we are given an iso $\mathrm{m}^0 \in \mathcal{L}(1, !\top)$ and a natural iso $\mathrm{m}^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \,\&\, Y))$ which satisfy a series of commutations that we do not record here. We also require a coherence condition relating m^2 and dig.

We use $?\_$ for the "De Morgan dual" of $!\_$: $?X = (!(X^\perp))^\perp$ and similarly for morphisms. It is a monad on $\mathcal{L}$ with unit der'_X and multiplication dig'_X defined straightforwardly, using der_Y and dig_Y.

2.1.1 Lax monoidality. It follows that we can define a lax symmetric monoidal structure for the functor $!\_$ from the symmetric monoidal category $(\mathcal{L}, \otimes)$ to itself. This means that we can define a morphism $\mu^0 \in \mathcal{L}(1, !1)$ and a natural transformation $\mu^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \otimes Y))$ which satisfy some coherence diagrams whose main consequence is that we can canonically extend this natural transformation to the case of n-ary tensors:

$$\mu^{(n)}_{X_1,\dots,X_N} \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, !(X_1 \otimes \cdots \otimes X_n))$$

in a way which is compatible with the symmetric monoidal structure of $\mathcal{L}$ (and allows us to write things just as if $\otimes$ were strictly associative).

2.1.2 The Eilenberg-Moore category. It is then standard to define the category $\mathcal{L}^!$ of !-coalgebras. An object of this category is a pair $P = (\underline{P}, \mathrm{h}_P)$ where $\underline{P} \in \mathsf{Obj}(\mathcal{L})$ and $\mathrm{h}_P \in \mathcal{L}(\underline{P}, !\underline{P})$ is such that $\mathrm{der}_{\underline{P}}\ \mathrm{h}_P = \mathrm{Id}$ and $\mathrm{dig}_{\underline{P}}\ \mathrm{h}_P = !\mathrm{h}_P\, \mathrm{h}_P$.

Given two such coalgebras P and Q, an element of $\mathcal{L}^!(P, Q)$ is an $f \in \mathcal{L}(\underline{P}, \underline{Q})$ such that $\mathrm{h}_Q\, f = !f\, \mathrm{h}_P$. Identities and composition are defined in the obvious way. The functor $!\_$ can then be seen as a functor from $\mathcal{L}$ to $\mathcal{L}^!$: this functor maps X to the coalgebra $(!X, \mathrm{dig}_X)$ and a morphism $f \in \mathcal{L}(X, Y)$ to the coalgebra morphism $!f \in \mathcal{L}^!((!X, \mathrm{dig}_X), (!Y, \mathrm{dig}_Y))$. This functor is right adjoint to the forgetful functor $\mathsf{U} : \mathcal{L}^! \to \mathcal{L}$ which maps a !-coalgebra P to $\underline{P}$ and a morphism f to itself. Given $f \in \mathcal{L}(\underline{P}, X)$, we use $f^! \in \mathcal{L}^!(P, !X)$ for the morphism associated with f by this adjunction. It is given by $f^! = !f\, \mathrm{h}_P$. Observe that, if $g \in \mathcal{L}^!(Q, P)$, we have

$$f^!\, g = (f\, g)^! \tag{3}$$

The object 1 of $\mathcal{L}$ induces an object of $\mathcal{L}^!$, still denoted as 1, namely $(1, \mu^0)$.

Given two objects P and Q of $\mathcal{L}^!$, we can define an object $P \otimes Q$ of $\mathcal{L}^!$ setting $\underline{P \otimes Q} = \underline{P} \otimes \underline{Q}$ and by defining $\mathrm{h}_{P\otimes Q}$ as the following composition of morphisms

$$\underline{P} \otimes \underline{Q} \xrightarrow{\mathrm{h}_P \otimes \mathrm{h}_Q} !\underline{P} \otimes !\underline{Q} \xrightarrow{\mu^2_{\underline{P},\underline{Q}}} !(\underline{P} \otimes \underline{Q})$$

Any object P of $\mathcal{L}^!$ can be equipped with a canonical structure of commutative comonoid. This means that we can define a morphism $\mathrm{w}_P \in \mathcal{L}^!(P, 1)$ and a morphism $\mathrm{c}_P \in \mathcal{L}^!(P, P \otimes P)$ which satisfy the commutations recorded in Figure 3.

One can check a stronger property, namely that 1 is the terminal object of $\mathcal{L}^!$ and that $P \otimes Q$ (equipped with projections defined in the obvious way using w_Q and w_P) is the cartesian product of P and Q in $\mathcal{L}^!$; the proof consists of surprisingly long computations for which we refer again to [22].

It is also important to notice that, if the family $(P_i)_{i\in I}$ of objects of $\mathcal{L}^!$ is such that the family $(\underline{P_i})_{i\in I}$ admits a coproduct $(\bigoplus_{i\in I} \underline{P_i}, (\mathrm{in}_i)_{i\in I})$ in $\mathcal{L}$, then it admits a coproduct in $\mathcal{L}^!$. This coproduct $P = \bigoplus_{i\in I} P_i$ is defined by $\underline{P} = \bigoplus_{i\in I} \underline{P_i}$, with a structure map h_P defined by the fact that, for each $i \in I$, $\mathrm{h}_P\, \mathrm{in}_i$ is the following composition of morphisms:

$$\underline{P_i} \xrightarrow{\mathrm{h}_{P_i}} !\underline{P_i} \xrightarrow{!\mathrm{in}_i} !\underline{P}$$

2.1.3 Fix-point operators.

For any object X, we assume to be given a morphism $\mathrm{fix}_X \in \mathcal{L}(!(!X \multimap X), X)$ such that the following diagram commutes

$$\begin{tikzcd}
!(!X \multimap X) \arrow[r, "\mathrm{c}_{!X}"] \arrow[ddr, "\mathrm{fix}_X"'] & !(!X \multimap X) \otimes !(!X \multimap X) \arrow[d, "\mathrm{der}_{!X \multimap X} \otimes \mathrm{fix}_X^!"] \\
& (!X \multimap X) \otimes !X \arrow[d, "\mathrm{ev}"] \\
& X
\end{tikzcd}$$

2.2 Embedding-retraction pairs

We introduce now the categorical assumptions that we use to interpret fix-points of types.

We assume that 0 and $\top$ are isomorphic; these isos being unique, we assume that 0 and $\top$ are the same objects[2].

We assume to be given a category $\mathcal{L}_\subseteq$ such that $\mathsf{Obj}(\mathcal{L}_\subseteq) = \mathsf{Obj}(\mathcal{L})$ together with a functor $\mathsf{F} : \mathcal{L}_\subseteq \to \mathcal{L}^{\mathsf{op}} \times \mathcal{L}$ such that $\mathsf{F}(X) = (X, X)$ and for which we use the notation $(\varphi^-, \varphi^+) = \mathsf{F}(\varphi)$. We assume that $\varphi^-\, \varphi^+ = \mathrm{Id}_X$. We define E (for *embedding*) as the functor $\mathrm{pr}_2\, \mathsf{F} : \mathcal{L}_\subseteq \to \mathcal{L}$. We assume moreover that the following properties hold.

- Given any countable filtered category J, any functor $\mathcal{D} : J \to \mathcal{L}_\subseteq$ has a colimit X in $\mathcal{L}_\subseteq$. Let $(\varphi_i \in \mathcal{L}_\subseteq(\mathcal{D}(i), X))_{i\in J}$ be the corresponding colimit cocone in $\mathcal{L}_\subseteq$. We assume moreover that $(\varphi_i^+ \in \mathcal{L}(\mathcal{D}(i), X))_{i\in J}$ is the colimit cocone of the functor $\mathsf{E}\mathcal{D}$ in the category $\mathcal{L}$. Concretely, this means that, given any cocone, that is, given any family of morphisms $(f_i \in \mathcal{L}(\mathcal{D}(i), Y))_{i\in J}$ such that, for any $l \in J(i, i')$ one has $f_{i'}\, \mathcal{D}(l)^+ = f_i$, there is exactly one morphism $f \in \mathcal{L}(X, Y)$ such that $f\, \varphi_i^+ = f_i$ for each $i \in J$.
- 0 is initial in $\mathcal{L}_\subseteq$ with $\mathsf{E}\theta^X = \mathsf{in}^X$ if θ^X is the unique element of $\mathcal{L}_\subseteq(0, X)$.
- There is a continuous functor[3] $\otimes_\subseteq : \mathcal{L}_\subseteq^2 \to \mathcal{L}_\subseteq$ which behaves as $\otimes$ on objects and satisfies $(\varphi \otimes_\subseteq \psi)^+ = \varphi^+ \otimes \psi^+$ and $(\varphi \otimes_\subseteq \psi)^- = \varphi^- \otimes \psi^-$. We use the same notation $\otimes$ for the functor $\otimes_\subseteq$. We make similar assumptions for $\oplus$ and $!\_$.
- There is a continuous functor $\mathcal{N} : \mathcal{L}_\subseteq \to \mathcal{L}_\subseteq$ such that $\mathcal{N}(X) = X^\perp$, $\mathcal{N}(\varphi)^+ = (\varphi^-)^\perp$ and $\mathcal{N}(\varphi)^- = (\varphi^+)^\perp$. We simply denote $\mathcal{N}(\varphi)$ as $\varphi^\perp$; remember that this operation is covariant.

So we can define a continuous covariant functor $\multimap : \mathcal{L}_\subseteq^2 \to \mathcal{L}_\subseteq$ by $X \multimap Y = (X \otimes Y^\perp)^\perp$ and $\varphi \multimap \psi = (\varphi \otimes \psi^\perp)^\perp$, so that $(\varphi \multimap \psi)^+ = \varphi^- \multimap \psi^+$ and $(\varphi \multimap \psi)^- = \varphi^+ \multimap \psi^-$ in $\mathcal{L}$.

We need to extend this notion of embedding-retraction pair to !-coalgebras because we want to define fix-points of positive types. Let $\mathcal{L}_\subseteq^!$ be the category whose objects are those of $\mathcal{L}^!$ and where

$$\mathcal{L}_\subseteq^!(P, Q) = \{\varphi \in \mathcal{L}_\subseteq(\underline{P}, \underline{Q}) \mid \varphi^+ \in \mathcal{L}^!(P, Q)\}\,.$$

In this definition, it is important *not to require* φ^- to be a coalgebra morphism. We still use U for the obvious forgetful functor $\mathcal{L}_\subseteq^! \to \mathcal{L}_\subseteq$. Observe that $\otimes$ and $\oplus$ define functors $(\mathcal{L}_\subseteq^!)^2 \to \mathcal{L}_\subseteq^!$ and that $!\_$ defines a functor $\mathcal{L}_\subseteq \to \mathcal{L}_\subseteq^!$.

Let J be a countable filtered category and let $\mathcal{E} : J \to \mathcal{L}_\subseteq^!$ be a functor. Let X be the colimit of the functor $\mathsf{U}\,\mathcal{E}$ in $\mathcal{L}_\subseteq$ and let $(\varphi_i \in \mathcal{L}_\subseteq(\mathsf{U}\mathcal{E}(i), X))_{i\in J}$ be the corresponding colimit cocone. We know that $(\varphi_i^+ \in \mathcal{L}(\mathsf{U}\mathcal{E}(i), X))_{i\in J}$ is a colimit cocone in $\mathcal{L}$. In particular, to prove that two morphisms $g, g' \in \mathcal{L}(X, Y)$ are equal, it suffices to prove that $g\, \varphi_i^+ = g'\, \varphi_i^+$ for each $i \in J$.

We want to equip X with a coalgebra structure $h \in \mathcal{L}(X, !X)$. For this, due to this universal property, it suffices to define a cocone $(f_i \in \mathcal{L}(\mathsf{U}\mathcal{E}(i), !X))_{i\in J}$. We set $f_i = !\varphi_i^+\, \mathrm{h}_{\mathcal{E}(i)}$ and h is completely characterized by the fact that $h\, \varphi_i^+ = f_i$ for each $i \in J$.

Let us prove that $\mathrm{der}_X\, h = \mathrm{Id}_X$. We have $\mathrm{der}_X\, h\, \varphi_i^+ = \mathrm{der}_X\, !\varphi_i^+\, \mathrm{h}_{\mathcal{E}(i)} = \varphi_i^+\, \mathrm{der}_{\underline{\mathcal{E}(i)}}\, h_{\mathcal{E}(i)} = \varphi_i^+$ and the result follows from the universal property. The equation $\mathrm{dig}_X\, h = !h\, h$ is proven similarly: we have $\mathrm{dig}_X\, h\, \varphi_i^+ = \mathrm{dig}_X\, !\varphi_i^+\, \mathrm{h}_{\mathcal{E}(i)} = !!\varphi_i^+\, \mathrm{dig}_{\underline{\mathcal{E}(i)}}\, \mathrm{h}_{\mathcal{E}(i)} = !!\varphi_i^+\, !\mathrm{h}_{\mathcal{E}(i)}\, \mathrm{h}_{\mathcal{E}(i)} = !f_i\, \mathrm{h}_{\mathcal{E}(i)}$ and $!h\, h\, \varphi_i^+ = !h\, !\varphi_i^+\, \mathrm{h}_{\mathcal{E}(i)} = !f_i\, \mathrm{h}_{\mathcal{E}(i)}$.

So we have proven that $\mathcal{E}$ has a colimit in the category $\mathcal{L}^!$ of coalgebras. A functor $\Phi : \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \to \mathcal{M}$ (where $\mathcal{M}$ and the $\mathcal{M}_i$s belong to $\{\mathcal{L}_\subseteq, \mathcal{L}_\subseteq^!\}$) is continuous if it commutes with countable filtered colimits.

Proposition 3 *The functors $\otimes$ and $\oplus$ from $(\mathcal{L}_\subseteq^!)^2$ to $\mathcal{L}_\subseteq^!$ are continuous. The functor $!\_ : \mathcal{L}_\subseteq \to \mathcal{L}_\subseteq^!$ is continuous. The functor $\multimap : (\mathcal{L}_\subseteq)^2 \to \mathcal{L}_\subseteq$ is continuous.*

This is an immediate consequence of our hypotheses and of the above considerations.

[2] This is true in many concrete models. It implies that any hom-set $\mathcal{L}(X, Y)$ has a distinguished element which coincides with the least element $\perp$ in denotational models based on domains or games. So this identification is typical of models featuring partial morphisms, which is required here because of the availability of fix-point operators for types and for programs.

[3] That is, a directed colimits preserving functor.

$$
\begin{tikzcd}
\underline{P} \arrow[r, "\mathrm{c}_P"] \arrow[dr, "\lambda_{\underline{P}}{}^{-1}"'] & \underline{P} \otimes \underline{P} \arrow[d, "\mathrm{w}_P \otimes \underline{P}"] \\
& 1 \otimes \underline{P}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "\mathrm{c}_P"] \arrow[d, "\mathrm{c}_P"'] & \underline{P} \otimes \underline{P} \arrow[r, "\mathrm{c}_P \otimes \underline{P}"] & (\underline{P} \otimes \underline{P}) \otimes \underline{P} \arrow[d, "\alpha_{\underline{P},\underline{P},\underline{P}}"] \\
\underline{P} \otimes \underline{P} \arrow[rr, "\underline{P} \otimes \mathrm{c}_P"'] & & \underline{P} \otimes (\underline{P} \otimes \underline{P})
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "\mathrm{c}_P"] \arrow[dr, "\mathrm{c}_P"'] & \underline{P} \otimes \underline{P} \arrow[d, "\sigma_{\underline{P},\underline{P}}"] \\
& \underline{P} \otimes \underline{P}
\end{tikzcd}
$$

Figure 3. Commutative $\otimes$-comonoid

Theorem 4 *Let $\Phi : (\mathcal{L}^!_\subseteq)^{n+1} \to \mathcal{L}^!_\subseteq$ be a continuous functor. There is a continuous functor Fix$(\Phi) : (\mathcal{L}^!_\subseteq)^n \to \mathcal{L}^!_\subseteq$ which is naturally isomorphic to the functor $\Psi : (\mathcal{L}^!_\subseteq)^n \to \mathcal{L}^!_\subseteq$ defined by $\Psi(P_1, \dots, P_n) = \Phi(P_1, \dots, P_n, \mathit{Fix}(\Phi)(P_1, \dots, P_n))$ (and similarly for morphisms).*

Proof. Let $\vec{P} = (P_1, \dots, P_n)$ be a tuple of objects of $\mathcal{L}^!$. Consider the functor $\Phi_{\vec{P}} : \mathcal{L}^!_\subseteq \to \mathcal{L}^!_\subseteq$ defined by $\Phi_{\vec{P}}(P) = \Phi(\vec{P}, P)$ and similarly for morphisms. Consider the set of natural numbers equipped with the usual order relation as a filtered category ($\mathbb{N}(n,m)$ has one element $l_{n,m}$ if $n \leq m$ and is empty otherwise). We define a functor $\mathcal{E} : \mathbb{N} \to \mathcal{L}^!_\subseteq$ as follows. First, we set $\mathcal{E}(i) = \Phi^i_{\vec{P}}(0)$. For each i, we define $\varphi_0 = \theta^{\underline{\mathcal{E}(1)}}$ and then we set $\varphi_{i+1} = \Phi_{\vec{P}}(\varphi_i)$. Then, given $i, j \in \mathbb{N}$ such that $i \leq j$, we set $\mathcal{E}(l_{i,j}) = \varphi_{j-1} \cdots \varphi_i$. We define $\mathrm{Fix}(\Phi)(\vec{P})$ as the colimit of this functor $\mathcal{E}$ in $\mathcal{L}^!_\subseteq$. By standard categorical methods using the universal property of colimits, we extend this operation to a continuous functor $\mathrm{Fix}(\Phi) : (\mathcal{L}^!_\subseteq)^n \to \mathcal{L}^!_\subseteq$ which satisfies the required condition by continuity of Φ. $\square$

2.3 Interpreting types and terms

With any positive type φ and any repetition-free list $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ of type variables containing all free variables of φ we associate a continuous functor $[\varphi]^!_{\vec{\zeta}} : (\mathcal{L}^!_\subseteq)^n \to \mathcal{L}^!_\subseteq$ and with any general type σ and any list $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ of pairwise distinct type variables containing all free variables of σ we associate a continuous functor $[\sigma]_{\vec{\zeta}} : (\mathcal{L}^!_\subseteq)^n \to \mathcal{L}_\subseteq$. We give the definition on objects, the definition on morphisms being similar.

$$
\begin{gathered}
[\zeta_i]^!_{\vec{\zeta}}(\vec{P}) = P_i \qquad [!\sigma]^!_{\vec{\zeta}}(\vec{P}) = !([\sigma]_{\vec{\zeta}}(\vec{P})) \\
[\varphi \otimes \psi]^!_{\vec{\zeta}}(\vec{P}) = [\varphi]^!_{\vec{\zeta}}(\vec{P}) \otimes [\psi]^!_{\vec{\zeta}}(\vec{P}) \\
[\varphi \oplus \psi]^!_{\vec{\zeta}}(\vec{P}) = [\varphi]^!_{\vec{\zeta}}(\vec{P}) \oplus [\psi]^!_{\vec{\zeta}}(\vec{P}) \\
[\mathsf{Fix}\, \zeta \cdot \varphi]^!_{\vec{\zeta}} = \mathrm{Fix}([\varphi]^!_{\vec{\zeta},\zeta}) \\
[\varphi]_{\vec{\zeta}} = \mathsf{U}\, [\varphi]^!_{\vec{\zeta}} \qquad [\varphi \multimap \sigma]_{\vec{\zeta}}(\vec{P}) = (\underline{[\varphi]^!{}_{\vec{\zeta}}(\vec{P})}) \multimap ([\sigma]_{\vec{\zeta}}(\vec{P}))
\end{gathered}
$$

When we write $[\sigma]$ or $[\varphi]^!$ (without subscript), we assume implicitely that the types σ and φ have no free type variables. Then $[\sigma]$ is an object of $\mathcal{L}$ and $[\varphi]^!$ is an object of $\mathcal{L}^!$.

Interpreting terms. Given a typing context $\mathcal{P} = (x_1 : \varphi_1, \dots, x_n : \varphi_n)$, we define $[\mathcal{P}]^!$ as the object $[\varphi_1]^! \otimes \cdots \otimes [\varphi_n]^!$ of $\mathcal{L}^!$. Notice that $\underline{[\mathcal{P}]^!} = [\varphi_1] \otimes \cdots \otimes [\varphi_n]$. We denote this object of $\mathcal{L}$ as $[\mathcal{P}]$.

Given a term M, a typing context $\mathcal{P} = (x_1 : \varphi_1, \dots, x_n : \varphi_n)$ and a type σ such that $\mathcal{P} \vdash M : \sigma$, we define $[M]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\sigma])$ by induction on the typing derivation of M (that is, on M).

Remark: A crucial observation is that $\mathcal{L}^!([\mathcal{P}]^!, [\varphi]^!) \subseteq \mathcal{L}([\mathcal{P}], [\varphi])$ for any positive type φ. Hence, for a term M such that $\mathcal{P} \vdash M : \varphi$, it may happen, *but it is not necessarily the case*, that $[M]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi]^!)$. The terms M which have this property are duplicable and discardable, and the main property of values is that they belong to this semantically defined class of terms. Let us call such terms *$\mathcal{L}$-central*, following the terminology of [13].

We define $[M]_\mathcal{P}$ by induction on the typing derivation, that is, on M.

If $M = x_i$ for $1 \leq i \leq n$, then $[M]_\mathcal{P} = \mathrm{pr}_i \in \mathcal{L}^!([\mathcal{P}]^!, [P_i]^!)$. Remember indeed that $[\mathcal{P}]^!$ is the cartesian product of $[P_1]^!, \dots, [P_n]^!$ in $\mathcal{L}^!$. Observe that M is $\mathcal{L}$-central.

Assume that $M = N^!$ and $\sigma = !\tau$ with $\mathcal{P} \vdash N : \tau$. By inductive hypothesis we have $[N]_\mathcal{P} \in \mathcal{L}(\underline{[\mathcal{P}]^!}, [\tau])$ and hence we can set $[M]_\mathcal{P} = [N]^!_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, ![\tau])$ so that M is $\mathcal{L}$-central.

Assume that $M = \langle M_1, M_2 \rangle$ and $\sigma = \varphi_1 \otimes \varphi_2$ with $\mathcal{P} \vdash M_i : \varphi_i$ for $i = 1, 2$. By inductive hypothesis we have defined $[M_i]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_i])$ for $i = 1, 2$. Since $[\mathcal{P}] = \underline{[\mathcal{P}]^!}$ we have a contraction morphism $\mathrm{c}_{[\mathcal{P}]^!} \in \mathcal{L}^!([\mathcal{P}]^!, [\mathcal{P}]^! \otimes [\mathcal{P}]^!)$ so that we can set $[M] = ([M_1]_\mathcal{P} \otimes [M_2]_\mathcal{P})\ \mathrm{c}_{[\mathcal{P}]^!} \in \mathcal{L}([\mathcal{P}], [\sigma])$. Hence if M_1 and M_2 are $\mathcal{L}$-central, then M is $\mathcal{L}$-central.

Assume that $M = \mathsf{in}_i N$ (for $i = 1$ or $i = 2$)) and $\sigma = \varphi_1 \oplus \varphi_2$. By inductive hypothesis we have $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_i])$ and since we have $\mathrm{in}_i \in \mathcal{L}^!([\varphi_i]^!, [\varphi_1]^! \oplus [\varphi_2]^!)$ it makes sense to set $[M]_\mathcal{P} = \mathrm{in}_i\, [N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\sigma])$. Observe that if N is $\mathcal{L}$-central then so is M.

Assume that $M = \lambda x^\varphi\, N$ and $\sigma = \varphi \multimap \tau$ with $\mathcal{P}, x : \varphi \vdash N : \tau$. By inductive hypothesis we have $[N]_{\mathcal{P},x:\varphi} \in \mathcal{L}([\mathcal{P}] \otimes [\varphi], [\tau])$ and we set $[M]_\mathcal{P} = \mathrm{cur}\,([N]_{\mathcal{P},x:\varphi}) \in \mathcal{L}([\mathcal{P}], [\varphi] \multimap [\tau])$. Of course, even if τ is positive and N is $\mathcal{L}$-central, M is not $\mathcal{L}$-central, simply because its type is not positive.

Assume that $M = \langle N \rangle R$ with $\mathcal{P} \vdash N : \varphi \multimap \sigma$ and $\mathcal{P} \vdash R : \varphi$ for some positive type φ. By inductive hypothesis we have $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi] \multimap [\sigma])$ and $[R]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi])$. Since $[\mathcal{P}] = \underline{[\mathcal{P}]^!}$ we have a contraction morphism $\mathrm{c}_{[\mathcal{P}]^!} \in \mathcal{L}^!([\mathcal{P}]^!, [\mathcal{P}]^! \otimes [\mathcal{P}]^!)$ so that we can set $[M]_\mathcal{P} = \mathrm{ev}\,([N]_\mathcal{P} \otimes [R]_\mathcal{P})\ \mathrm{c}_{[\mathcal{P}]^!} \in \mathcal{L}([\mathcal{P}], [\sigma])$.

Assume that $M = \mathsf{case}(N, (x_1)R_1, (x_2)R_2)$ with $\mathcal{P} \vdash N : \varphi_1 \oplus \varphi_2$ and $\mathcal{P}, x_i : \varphi_i \vdash R_i : \sigma$ for $i = 1, 2$. By inductive hypothesis we have $[M]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_1] \oplus [\varphi_2])$ and $[R_i]_\mathcal{P} \in \mathcal{L}([\mathcal{P}] \otimes [\varphi_i], [\sigma])$ for $i = 1, 2$. By the universal property of the coproduct $\oplus$ in $\mathcal{L}$ and by the fact that the functor $[\mathcal{P}] \otimes \_$ is a left adjoint, there is exactly one morphism $f \in \mathcal{L}([\mathcal{P}] \otimes ([\varphi_1] \oplus [\varphi_2]), [\sigma])$ such that $f\,([\mathcal{P}] \otimes \mathrm{in}_i) = [R_i]_\mathcal{P}$ for $i = 1, 2$. Then we set $[M]_\mathcal{P} = f\,([\mathcal{P}] \otimes [N]_\mathcal{P})\ \mathrm{c}_{[\mathcal{P}]^!}$.

Assume that $M = \mathsf{pr}_i N$ and $\sigma = \varphi_i$ for $i = 1$ or $i = 2$, with $\mathcal{P} \vdash N : \varphi_1 \otimes \varphi_2$. By inductive hypothesis we have $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_1] \otimes [\varphi_2])$. Then remember that we have the projection $\mathrm{pr}_i \in \mathcal{L}^!([\varphi_1]^! \otimes [\varphi_2]^!, [\varphi_i]^!)$ so that we can set $[M]_\mathcal{P} = \mathrm{pr}_i\, [N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_i])$.

Assume that $M = \mathsf{der}(N)$ with $\mathcal{P} \vdash N : !\sigma$. Then we have $\mathrm{der}_{[\sigma]} \in \mathcal{L}(![\sigma], [\sigma])$ so that we can set $[M]_\mathcal{P} = \mathrm{der}_{[\sigma]}\, [N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\sigma])$.

Assume that $M = \mathsf{fix}\, x^{!\sigma}\, N$ so that $\mathcal{P}, x : !\sigma \vdash N : \sigma$. By inductive hypothesis we have $\mathrm{cur}([N]_{\mathcal{P},x:!\sigma}) \in \mathcal{L}(\underline{[\mathcal{P}]^!}, ![\sigma] \multimap [\sigma])$ and hence $(\mathrm{cur}([N]_{\mathcal{P},x:!\sigma}))^! \in \mathcal{L}^!([\mathcal{P}], !(\underline{![\sigma] \multimap [\sigma]}))$ so that we can set $[M]_\mathcal{P} = \mathrm{fix}\,(\mathrm{cur}([N]_{\mathcal{P},x:!\sigma}))^! \in \mathcal{L}([\mathcal{P}], [\sigma])$.

Proposition 5 *If $\mathcal{P} \vdash V : \varphi$ and V is a value, then V is $\mathcal{L}$-central, that is $[V]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi]^!)$.*

The proof is a straightforward verification (in the definition of the interpretation of terms we have singled out the constructions which preserve $\mathcal{L}$-centrality). The main operational feature of $\mathcal{L}$-central terms is that they enjoy the following substitivity property.

Proposition 6 (Substitution Lemma) *Assume that $\mathcal{P}, x : \varphi \vdash M : \sigma$ and $\mathcal{P} \vdash N : \varphi$, and assume that N is $\mathcal{L}$-central. Then we have*

$$[M\,[N/x]]_{\mathcal{P}} = [M]_{\mathcal{P},x:\varphi}\,([\mathcal{P}] \otimes [N]_{\mathcal{P}})\ \mathrm{c}_{[\mathcal{P}]^!}$$

Proof. Induction on M using in an essential way the $\mathcal{L}$-centrality of N. Let us consider two cases to illustrate this point. Assume first that $M = R^!$ and $\sigma = {!\tau}$, with $\mathcal{P}, x : \varphi \vdash R : \tau$ so that $[R]_{\mathcal{P},x:\varphi} \in \mathcal{L}([\mathcal{P}] \otimes [\varphi], [\sigma])$. Then we have $[M\,[N/x]]_{\mathcal{P}} = [R\,[N/x]^!]_{\mathcal{P}} = ([R\,[N/x]]_{\mathcal{P}})^! = ([R]_{\mathcal{P},x:\varphi}\,([\mathcal{P}] \otimes [N]_{\mathcal{P}})\ \mathrm{c}_{[\mathcal{P}]})^!$ by inductive hypothesis. We obtain the contended equation by applying Equation (3) and the fact that $[\mathcal{P}] \otimes [N]_{\mathcal{P}}$ is a coalgebra morphism since N is $\mathcal{L}$-central, and the fact that $\mathrm{c}_{[\mathcal{P}]}$ is also a coalgebra morphism.

Let us also consider the case $M = \langle M_1, M_2\rangle$ and $\sigma = \varphi_1 \otimes \varphi_2$ with $\mathcal{P}, x : \varphi \vdash M_i : \varphi_i$ for $i = 1, 2$ so that $[M_i]_{\mathcal{P},x:\varphi} \in \mathcal{L}([\mathcal{P}] \otimes [\varphi], [\varphi_i])$ for $i = 1, 2$. We have

$$\begin{aligned}
[M\,[N/x]]_{\mathcal{P}} &= [\langle M_1\,[N/x]\,, M_2\,[N/x]\rangle]_{\mathcal{P}}\\
&= ([M_1\,[N/x]]_{\mathcal{P}} \otimes [M_2\,[N/x]]_{\mathcal{P}})\ \mathrm{c}_{[\mathcal{P}]^! \otimes [\varphi]^!}\\
&= (([M_1]_{\mathcal{P},x:\varphi}\,([\mathcal{P}] \otimes [N]_{\mathcal{P}})) \otimes ([M_2]_{\mathcal{P},x:\varphi}\,([\mathcal{P}] \otimes [N]_{\mathcal{P}})))\\
&\qquad \mathrm{c}_{[\mathcal{P}]^! \otimes [\varphi]^!}\\
&= ([M_1]_{\mathcal{P}} \otimes [M_2]_{\mathcal{P}})\ \mathrm{c}_{[\mathcal{P}]^!}\ ([\mathcal{P}] \otimes [N]_{\mathcal{P}})
\end{aligned}$$

using the inductive hypothesis, naturality of contraction in $\mathcal{L}^!$ and the fact that $([\mathcal{P}] \otimes [N]_{\mathcal{P}})$ is a coalgebra morphism since N is $\mathcal{L}$-central. The other cases are handled similarly. □

Theorem 7 (Soundness) *If $\mathcal{P} \vdash M : \sigma$ and $M \to_{\mathsf{w}} M'$ then $[M]_{\mathcal{P}} = [M']_{\mathcal{P}}$.*

Proof. By induction on the derivation that $M \to_{\mathsf{w}} M'$, using the Substitution Lemma and the $\mathcal{L}$-centrality of values. □

3. Scott semantics

Usually, in a model $\mathcal{L}$ of LL, an object X of $\mathcal{L}$ can be endowed with several different structures of !-coalgebras which makes the category $\mathcal{L}^!$ difficult to describe simply (in contrast with the Kleisli category used for interpreting PCF; its objects are those of $\mathcal{L}$). In the Scott model of LL (see *eg.* [9]) however, every object of the linear category has exactly one structure of !-coalgebra as we shall see now. This is certainly a distinctive feature of this model. Such a property does not hold for instance in coherence spaces. A nice outcome of these observations will be a very simple intersection typing system for CbPV.

3.1 The Scott semantics of LL

We introduce a "linear" category $\mathbf{Polr}$ of preorders and relations. A preorder is a pair $S = (|S|, \leq_S)$ where $|S|$ is an at most countable set and $\leq_S$ (written $\leq$ when no confusion is possible) is a preorder relation on $|S|$. Given two preorders S and T, a morphism from S to T is a $f \subseteq |S| \times |T|$ such that, if $(a, b) \in f$ and $(a', b') \in |S| \times |T|$ satisfy $a \leq_S a'$ and $b' \leq_S b$, then $(a', b') \in f$. The relational composition of two morphisms is still a morphism and the identity morphism at S is $\mathrm{Id}_S = \{(a, a') \mid a' \leq_S a\}$.

Given an object S in $\mathbf{Polr}$, the set $\mathsf{Ini}(S)$ of downwards closed subsets of $|S|$, ordered by inclusion, is a complete lattice which is ω-prime-algebraic (and all such lattices are of that shape up to iso). $\mathbf{Polr}$ is equivalent to the category of ω-prime algebraic complete lattices and linear maps (functions preserving all lubs).

3.1.1 Monoidal structure and cartesian product.

The object 1 is $(\{*\}, =)$ and given preorders S and T we set $S \otimes T = (|S| \times |T|, \leq_S \times \leq_T)$. The tensor product of morphisms is defined in the obvious way. The isos defining the monoidal structure are easy to define. Then one defines $S \multimap T$ by $|S \multimap T| = |S| \times |T|$ and $(a', b') \leq_{S \multimap T} (a, b))$ if $a \leq a'$ and $b' \leq b$. The linear evaluation morphism $\mathrm{ev} \in \mathbf{Polr}((S \multimap T) \otimes S, T)$ is given by $\mathrm{ev} = \{(((a', b), a), b') \mid b' \leq b \text{ and } a' \leq a\}$. If $f \in \mathbf{Polr}(U \otimes S, T)$ then $\mathrm{cur}(f) \in \mathbf{Polr}(U, S \multimap T)$ is defined by moving parentheses. This shows that $\mathbf{Polr}$ is closed. It is *-autonomous, with $\perp = 1$ as dualizing object. Observe that $S^\perp$ is simply $|S|$ equipped with $\geq_S$ as preorder relation $\leq_{S^\perp}$.

Given a countable family of objects $(S_i)_{i \in I}$, the cartesian product S is defined by $|S| = \bigcup_{i \in I} \{i\} \times |S_i|$ with $(i, a) \leq (j, b)$ if $i = j$ and $a \leq b$. Projections are defined by $\mathrm{pr}_i = \{((i, a), a') \mid a' \leq a\}$. Tupling of morphisms is defined as in $\mathbf{Rel}$. Coproducts are defined similarly.

3.1.2 Exponential.

One sets $!S = (\mathcal{P}_{\mathrm{fin}}(|S|), \leq)$ with $u \leq u'$ if $\forall a \in u\, \exists a' \in u'\ a \leq_S a'$ (where $\mathcal{P}_{\mathrm{fin}}(E)$ is the set of all finite subsets of E. Given $f \in \mathbf{Polr}(S, T)$, one defines $!f$ as $\{(u, v) \in |!S| \times |!T| \mid \forall b \in v\, \exists a \in u\ (a, b) \in f\}$. It is easy to prove that this defines a functor $\mathbf{Polr} \to \mathbf{Polr}$. Then one sets $\mathrm{der}_S = \{(u, a) \mid \exists a' \in u\ a \leq a'\} \in \mathbf{Polr}(!S, S)$ and $\mathrm{dig}_S = \{(u, \{u_1, \dots, u_n\}) \mid u_1 \cup \cdots \cup u_n \leq_{!S} u\} \in \mathbf{Polr}(!S, !!S)$. This defines a comonad $\mathbf{Polr} \to \mathbf{Polr}$. The Seely isos are given by $\mathrm{m}^0 = \{(*, \emptyset)\} \in \mathbf{Polr}(1, !\top)$ and $\mathrm{m}^2_{S,T} = \{((u, v), w) \mid w \leq_{!(S \& T)} \{1\} \times u \cup \{2\} \times v\} \in \mathbf{Polr}(!S \otimes !T, !(S \& T))$.

Each object S has a fix-point operator $\mathrm{fix}_S \in \mathbf{Polr}(!(!S \multimap S), S)$ which is defined as a least fix-point: $\mathrm{fix}_S = \{(w, a) \mid \exists (u', a') \in w\ a \leq a' \text{ and } \forall a'' \in u'\ (w, a'') \in \mathrm{fix}_S\}$.

3.2 The category of !-coalgebras

The first main observation is that each object of $\mathbf{Polr}$ has exactly one structure of !-coalgebra.

Theorem 8 *Let S be an object of $\mathbf{Polr}$. Then (S, p_S) is a !-coalgebra, where $\mathrm{p}_S = \{(a, u) \in |S| \times |!S| \mid \forall a' \in u\ a' \leq a\}$. Moreover, if P is a !-coalgebra, then $\mathrm{h}_P = \mathrm{p}_{\underline{P}}$.*

The proof is easy and is provided as complementary material.

3.2.1 Morphisms of !-coalgebras.

Now that we know that the objects of $\mathbf{Polr}^!$ are those of $\mathbf{Polr}$, we turn our attention to morphisms. When we consider a preorder S as an object of $\mathbf{Polr}^!$, we always mean the object (S, p_S) described above.

With any preorder S, we have associated an ω-prime algebraic complete lattice $\mathsf{Ini}(S)$. We associate now with such a preorder an ω-algebraic cpo $\mathsf{Idl}(S)$ which is the ideal completion of S: an element of $\mathsf{Idl}(S)$ is a subset ξ of $|S|$ such that ξ is non-empty, downwards closed and directed (meaning that if $a, a' \in \xi$ then there is $a'' \in \xi$ such that $a, a' \leq_S a''$). We equip $\mathsf{Idl}(S)$ with the inclusion partial order relation.

Lemma 9 *For any preorder S, the partially ordered set $\mathsf{Idl}(S)$ is a cpo which has countably many isolated elements. Moreover, for any $\xi \in \mathsf{Idl}(S)$, the set of isolated elements $\xi_0 \in \mathsf{Idl}(S)$ such that $\xi_0 \subseteq \xi$ is directed and ξ is the lub of that set. In general, $\mathsf{Idl}(S)$ has no minimum element.*

The proof is straightforward, the isolated elements are the $\downarrow a$ for $a \in |S|$. Such a cpo can be called an *ω-algebraic predomain*.

It is not necessarily bounded-complete and has not necessarily a minimum element.

Theorem 10 *Given preorders S and T, there is a bijective and functorial correspondence between $\mathbf{Polr}^!(S,T)$ and the set of Scott continuous functions from $\mathsf{Idl}(S)$ to $\mathsf{Idl}(T)$. Moreover, this correspondence is an order isomorphism when Scott-continuous functions are equipped with the usual pointwise ordering relation and $\mathbf{Polr}^!(S,T)$ is equipped with the inclusion order on relations.*

The proof is provided as complementary material.

Let $\mathbf{Predom}$ be the category whose objects are the preorders and where a morphism from S to T is a Scott continuous function from $\mathsf{Idl}(S)$ to $\mathsf{Idl}(T)$. We have seen that $\mathbf{Polr}^!$ and $\mathbf{Predom}$ are equivalent categories (isomorphic indeed). It is easy to retrieve directly the fact that $\mathbf{Predom}$ has products and sums: the product of S and T is $S \otimes T$ (and indeed, it is easy to check that $\mathsf{Idl}(S \otimes T) \simeq \mathsf{Idl}(S) \times \mathsf{Idl}(T)$) and their sum is $S \oplus T$ and indeed $\mathsf{Idl}(S \oplus T) \simeq \mathsf{Idl}(S) + \mathsf{Idl}(T)$, the disjoint union of the predomains $\mathsf{Idl}(S)$ and $\mathsf{Idl}(T)$. This predomain has no minimum element as soon as $|S|$ and $|T|$ are non-empty. Observe also that $\mathsf{Idl}(!S) = \mathsf{Ini}(S)$: one retrieves the fact that the Kleisli category of the ! comonad is the category of preorders and Scott continuous functions between the associated lattices.

3.2.2 Inclusions and embedding-retraction pairs.

We define a category $\mathbf{Polr}_\subseteq$ as follows: the objects are those of $\mathbf{Polr}$ and $\mathbf{Polr}_\subseteq(S,T)$ is a singleton $\{\varphi_{S,T}\}$ if $|S| \subseteq |T|$ and $\forall a, a' \in |S|\ a \leq_S a' \Leftrightarrow a \leq_T a'$ (and then we write $S \subseteq T$) and is empty otherwise. So $(\mathbf{Polr}_\subseteq, \subseteq)$ is a partially ordered class. The functor F is then defined as follows: if $S \subseteq T$ then $\varphi^+_{S,T} = \{(a,b) \in |S| \times |T| \mid b \leq_T a\}$ and $\varphi^-_{S,T} = \{(b,a) \in |T| \times |S| \mid a \leq_T b\}$; this definition is functorial and $\varphi^-_{S,T}\, \varphi^+_{S,T} = \mathrm{Id}_S$. The partially ordered class $\mathbf{Polr}_\subseteq$ is complete in the sense that any directed family of objects[4] $(S_i)_{i\in J}$ has a lub S given by $|S| = \cup_{i\in J}|S_i|$ and $a \leq_S a'$ if $a \leq_{S_i} a'$ for some i; we denote this preorder as $\cup_{i\in J} S_i$. The operations $\otimes$, $\oplus$ and $!\_$ are monotone and Scott-continuous operations on this partially ordered class.

Lemma 11 *Assume that $S_1 \subseteq S_2$ and let $f_i \in \mathbf{Polr}(S_i, T)$ for $i = 1, 2$. Then $f_1 = f_2\, \varphi^+_{S_1,S_2}$ iff $f_1 = f_2 \cap |S_1 \multimap T|$.*

The proof is provided as complementary material.

Given a directed family $(S_i)_{i\in J}$ in $\mathbf{Polr}_\subseteq$ and setting $S = \cup_{i\in J} S_i$, one proves easily using Lemma 11 that the cone $(\varphi^+_{S_i,S} \in \mathbf{Polr}(S_i, S))_{i\in J}$ is a colimit cone in $\mathbf{Polr}$. Consider indeed a family of morphisms $(f_i \in \mathbf{Polr}(S_i,T))_{i\in J}$ such that $i \leq j \Rightarrow f_i = f_j\, \varphi^+_{S_i,S_j}$, that is $f_i = f_j \cap |S_i \multimap T|$. Then $f = \cup_{i\in J} f_i$ is the unique element of $\mathbf{Polr}(S,T)$ such that $f\, \varphi^+_{S_i,S} = f_i$ for each $i \in J$. So the category $\mathbf{Polr}_\subseteq$ satisfies all the axioms of Section 2.2.

As explained in that section, this allows to define fixpoints of positive types. As a first example consider the type of flat natural numbers $\iota = \mathsf{Fix}\,\zeta \cdot (1 \oplus \zeta)$ where $1 = !\top$, so that $|1| = \{\emptyset\}$. Up to renaming we have $|[\iota]| = \mathbb{N}$ and $n \leq_{[\iota]} n'$ iff $n = n'$. The coalgebraic structure of this positive type is given by $\mathrm{h}_{[\iota]} = \{(n, \emptyset) \mid n \in \mathbb{N}\} \cup \{(n, \{n\}) \mid n \in \mathbb{N}\}$. Consider now the type $\rho = \mathsf{Fix}\,\zeta \cdot (1 \oplus (\iota \otimes !\zeta))$ of lazy lists of flat natural numbers. The interpretation S of this type is the least fix-point of the continuous functor (that is, the Scott continuous functional) $[1 \oplus (\iota \otimes !\zeta)]_\zeta : \mathbf{Polr}_\subseteq \to \mathbf{Polr}_\subseteq$. So $|S| = \cup_{n=0}^\infty U_n$ where $(U_n)_{n\in\mathbb{N}}$ is the monotone sequence of sets defined by $U_0 = \emptyset$ and $U_{n+1} = \{\emptyset\} \cup (\mathbb{N} \times \mathcal{P}_{\mathrm{fin}}(U_n))$ (this is a disjoint union). The preorder relation on $|S|$ is given by: $\emptyset \leq_S a$ iff $a = \emptyset$ and $(n,u) \leq_S a$ iff $a = (n,u')$ and $\forall b \in u \exists b' \in u'\ b \leq_S b'$. This preorder relation defines the coalgebraic structure of this positive type.

3.2.3 Well-foundedness of points.

Due to the possibility of using fix-points in the definition of types, the structural notion of subtype does not induce a well founded structure and therefore does not allow to perform proofs by induction. Nevertheless, the inductive definition of these fix-points induces a well-founded structure on the points of these types.

More precisely, we define a *predecessor* relation on pairs (σ, a) where σ is a type and $a \in [\sigma]$. We say that $(\sigma, a) \prec (\tau, b)$ if one of the following conditions holds.

- $\tau = !\sigma$, $b = u$ and $a \in u$.
- $\tau = \sigma_1 \otimes \sigma_2$, $b = (a_1, a_2)$ and $\sigma = \sigma_i$ and $a = a_i$ for $i = 1$ or $i = 2$.
- $\tau = \sigma_1 \oplus \sigma_2$, $b = (i, a)$ and $\sigma = \sigma_i$ for $i = 1$ or $i = 2$.
- $\tau = \varphi \multimap \sigma'$, $b = (c, a')$ and $\sigma = \varphi$ and $a = c$, or $\sigma = \sigma'$ and $a = a'$.

It is easily checked that there are no infinite sequence $(\sigma_i, a_i)_{i\in\mathbb{N}}$ such that $(\sigma_{i+1}, a_{i+1}) \prec (\sigma_i, a_i)$ for all $i \in \mathbb{N}$.

3.3 Scott semantics as a typing system

It is interesting to present the Scott semantics of terms as a typing system, in the spirit of Coppo-Dezani Intersection Types, see [15]. A *semantic context* is a sequence $\Phi = (x_1 : a_1 : \varphi_1, \ldots, x_n : a_n : \varphi_n)$ where $a_i \in [\varphi_i]$ for each i, its *underlying typing context* is $\underline{\Phi} = (x_1 : \varphi_1, \ldots, x_n : \varphi_n)$ and its *underlying tuple* is $\langle\Phi\rangle = (a_1, \ldots, a_k) \in [\underline{\Phi}]$. The typing rules are given in Figure 4.

A simple induction on typing derivation trees shows that this typing system in "monotone" as usually for intersection type systems. We write $\Phi \leq \Phi'$ if $\Phi = (x_1 : a_1 : \varphi_1, \ldots, x_n : a_n : \varphi_n)$, $\Phi' = (x_1 : a'_1 : \varphi_1, \ldots, x_n : a'_n : \varphi_n)$ and $a_i \leq_{[\varphi_i]} a'_i$ for $i = 1, \ldots, n$.

Proposition 12 *If $\Phi \vdash M : a : \sigma$, $a' \leq_{[\sigma]} a$ and $\Phi \leq \Phi'$ then $\Phi' \vdash M : a' : \sigma$.*

Using this property, one can prove that this deduction system describes exactly the Scott denotational semantics of CbPV.

Proposition 13 *Given $a_1 \in [\varphi_1]$,…,$a_n \in [\varphi_n]$ and $a \in [\sigma]$, one has $(a_1, \ldots, a_n, a) \in [M]_{x_1:\sigma_n,\ldots,x_1:\sigma_n}$ iff $x_1 : a_1 : \varphi_1, \ldots, x_n : a_n : \varphi_n \vdash M : a : \sigma$.*

The proof also uses crucially the fact that all structural operations (weakening, contraction, dereliction, promotion) admit a very simple description in terms of the preorder relation on objects thanks to Theorem 8; for instance the contraction morphism of an object S (seen as an object of $\mathbf{Polr}^!$) is $\mathrm{c}_S = \{(a, (a_1, a_2)) \mid a_i \leq a \text{ for } i = 1, 2\}$.

3.4 Adequacy

Our goal now is to prove that, if a closed term M of positive type φ has a non-empty interpretation, that is, if there is $a \in |[\varphi]|$ such that $\vdash M : a : \varphi$, then the reduction $\to_{\mathsf{w}}$ starting from M terminates. We use a semantic method adapted *eg.* from the presentation of the *reducibility* method in [15].

Given a type σ and an $a \in [\sigma]$, we define a set $|a|^\sigma$ of terms M such that $\vdash M : \sigma$ (so these terms are all closed). The definition

[4] Because we are dealing with a partially ordered class, we can replace general filtered categories with directed posets.

$$\frac{a' \leq_{[\varphi]} a}{\Phi, x : a : \varphi \vdash x : a' : \varphi} \quad \frac{u \in \mathcal{P}_{\text{fin}}(|[\sigma]|) \quad \forall a \in u \quad \Phi \vdash M : a : \sigma}{\Phi \vdash M^! : u : !\sigma} \quad \frac{\Phi \vdash M_1 : a_1 : \varphi_1 \quad \Phi \vdash M_2 : a_2 : \varphi_2}{\Phi \vdash \langle M_1, M_2 \rangle : (a_1, a_2) : \varphi_1 \otimes \varphi_2}$$

$$\frac{\Phi \vdash M : a : \varphi_i}{\Phi \vdash \mathsf{in}_i M : (i, a) : \varphi_1 \oplus \varphi_2} \quad \frac{\Phi \vdash M : (a, b) : \varphi \multimap \sigma \quad \Phi \vdash N : a : \varphi}{\Phi \vdash \langle M \rangle N : b : \sigma}$$

$$\frac{\Phi \vdash M : (1, a) : \varphi_1 \oplus \varphi_2 \quad \Phi, x_1 : a : \varphi_1 \vdash N_1 : b : \sigma \quad \underline{\Phi}, x_2 : \varphi_2 \vdash N_2 : \sigma}{\Phi \vdash \mathsf{case}(M, (x_1)N_1, (x_2)N_2) : b : \sigma}$$

$$\frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2}{\Phi \vdash \mathsf{pr}_i M : a_i : \varphi_i} \quad \frac{\Phi \vdash M : \{a\} : !\sigma}{\Phi \vdash \mathsf{der}(M) : a : \sigma} \quad \frac{\Phi, x : u : !\sigma \vdash M : a : \sigma \quad \forall b \in u \quad \Phi \vdash \mathsf{fix}\, x^{!\sigma}\, M : b : \sigma}{\Phi \vdash \mathsf{fix}\, x^{!\sigma}\, M : a : \sigma}$$

Figure 4. Scott Semantics as a Typing System

$$
\begin{aligned}
|u|_{\mathsf{v}}^{!\sigma} &= \{N^! \mid N \in \bigcap_{a \in u}^{n} |a|^{\sigma}\} \\
|(a_1, a_2)|_{\mathsf{v}}^{\varphi_1 \otimes \varphi_2} &= \{\langle V_1, V_2 \rangle \mid V_i \in |a_i|_{\mathsf{v}}^{\varphi_i} \text{ for } i = 1, 2\} \\
|(i, a)|_{\mathsf{v}}^{\varphi_1 \oplus \varphi_2} &= \{\mathsf{in}_i V \mid V \in |a|_{\mathsf{v}}^{\varphi_i}\} \\
|a|^{\varphi} &= \{M \mid \vdash M : \varphi \text{ and } \exists V \in |a|_{\mathsf{v}}^{\varphi}\ M \to_{\mathsf{w}}^* V\} \\
|(a, b)|^{\varphi \multimap \sigma} &= \{M \mid \vdash M : \varphi \multimap \sigma \text{ and } \forall V \in |a|_{\mathsf{v}}^{\varphi}\ \langle M \rangle V \in |b|^{\sigma}\}
\end{aligned}
$$

Figure 5. Interpretation of points as sets of terms in CbPV

is by induction on the point a (and not on the type σ, whose definition involves fix-points and is therefore not well-founded in general), or more precisely, on the pair (σ, a) using the well founded predecessor relation $\prec$ of Section 3.2.3.

Given a positive type φ and $a \in [\varphi]$, we define $|a|_{\mathsf{v}}^{\varphi}$ as a set of *closed values* V such that $\vdash V : \varphi$ and given a general type σ and $a \in [\sigma]$, we define $|a|^{\sigma}$ as a set of closed terms M such that $\vdash M : \sigma$. The definitions are by mutual induction and are given in Figure 5. Observe that for a value V such that $\vdash V : \varphi$ and for $a \in [\varphi]$, the statements $V \in |a|^{\varphi}$ and $V \in |a|_{\mathsf{v}}^{\varphi}$ are equivalent because V is normal for the $\to_{\mathsf{w}}$ reduction.

Lemma 14 *If $M \to_{\mathsf{w}} M' \in |a|^{\sigma}$ then $M \in |a|^{\sigma}$.*

Proof. By induction on the predecessor relation $\prec$. If $\sigma = \varphi$, the property follows readily from the definition. Assume that $\sigma = \varphi \multimap \tau$ and $a = (b, c)$. Assume that $M \to_{\mathsf{w}} M' \in |a|^{\sigma}$. Let $V \in |b|_{\mathsf{v}}^{\varphi}$, we have $\langle M \rangle V \to_{\mathsf{w}} \langle M' \rangle V$ and $\langle M' \rangle V \in |c|^{\tau}$ by definition of $|\varphi \multimap \tau|^{(b,c)}$. The announced property follows by inductive hypothesis. □

Lemma 15 *Let σ be a type and let $a, a' \in |[\sigma]|$ be such that $a \leq_{[\sigma]} a'$. Then $|a|^{\sigma} \supseteq |a'|^{\sigma}$. If σ is positive, we have $|a|_{\mathsf{v}}^{\varphi} \supseteq |a'|_{\mathsf{v}}^{\sigma}$*

Proof. We prove both statements by mutual induction on the $\prec$ relation.

Assume first that σ is a positive type that we prefer to denote as φ. Assume that $\varphi = !\tau$ so that $a, a' \in \mathcal{P}_{\text{fin}}(|[\tau]|)$ and $\forall b \in a\, \exists b' \in a'\ b \leq_{[\tau]} b'$. Let $V \in |a'|_{\mathsf{v}}^{\varphi}$ so that $V = N^!$ where $N \in \bigcap_{b' \in a'} |b'|^{\tau}$. Let $b \in a$. Let $b' \in a'$ be such that $b \leq_{[\tau]} b'$, we have $N \in |b'|^{\tau} \subseteq |b|^{\tau}$ by inductive hypothesis, and hence $N \in \bigcap_{b \in a} |b|^{\tau}$, so $V \in |a|_{\mathsf{v}}^{\varphi}$. Let now $M' \in |a'|^{\varphi}$, we know that there is $V' \in |a'|_{\mathsf{v}}^{\varphi}$ such that $M' \to^* V'$. We have just seen that $|a'|_{\mathsf{v}}^{\varphi} \subseteq |a|_{\mathsf{v}}^{\varphi}$ so $V' \in |a|_{\mathsf{v}}^{\varphi}$ and therefore $M' \in |a|^{\varphi}$. Assume that $\varphi = \varphi_1 \otimes \varphi_2$ (again, φ is positive) so that $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$ with $a_i \leq_{[\varphi_i]} a'_i$ for $i = 1, 2$. If $V' \in |a'|_{\mathsf{v}}^{\varphi}$ then $V' = \langle V'_1, V'_2 \rangle$ with $V'_i \in |a'_i|_{\mathsf{v}}^{\varphi_i}$ for $i = 1, 2$. By inductive hypothesis $V'_i \in |a_i|_{\mathsf{v}}^{\varphi_i}$ and hence $V \in |a|_{\mathsf{v}}^{\varphi}$. Just as above one proves that $|a'|^{\varphi} \subseteq |a|^{\varphi}$. The case where $\varphi = \varphi_1 \oplus \varphi_2$ is similar.

Assume last that $\sigma = \varphi \multimap \tau$ so that $a = (b, c)$, $a' = (b', c')$ with $b' \leq_{[\varphi]} b$ and $c \leq_{[\tau]} c'$. Let $M' \in |a'|^{\sigma}$, we have to prove that $M' \in |a|^{\sigma}$. Let therefore $V \in |b|_{\mathsf{v}}^{\varphi}$. By inductive hypothesis we have $V \in |b'|_{\mathsf{v}}^{\varphi}$ and therefore $\langle M' \rangle V \in |c'|^{\tau}$ so that $\langle M' \rangle V \in |c|^{\tau}$ by inductive hypothesis again. □

Theorem 16 *Let $\Phi = (x_1 : a_1 : \varphi_1, \dots, x_k : a_k : \varphi_k)$ and assume that $\Phi \vdash M : a : \sigma$. Then for any family of closed values $(V_i)_{i=1}^k$ such that $V_i \in |a_i|^{\varphi_i}$ one has $M\,[V_1/x_1, \dots, V_k/x_k] \in |a|^{\sigma}$.*

Proof. By induction on M. Let V_i be values such that $V_i \in |a_i|^{\varphi_i}$ for $i = 1, \dots, k$. For any term R, we use R' for $R\,[V_1/x_1, \dots, V_k/x_k]$. We use the definition of $\to_{\mathsf{w}}$, see Figure 2.

Assume first that $M = x_i$ for some $i \in \{1, \dots, k\}$; we know that $a \leq_{[\sigma_i]} a_i$. Then $M' = V_i$ and we have that $M' \in |a|^{\sigma}$ by Lemma 15.

Assume that $M = N^!$ with $\sigma = !\tau$, $a = u \in \mathcal{P}_{\text{fin}}(|[\tau]|)$, $\Phi \vdash N : b : \tau$ for each $b \in u$. By inductive hypothesis, we have $N' \in \bigcap_{b \in u} |b|^{\tau}$. Since $M' = N'^!$, and hence $M' \to_{\mathsf{w}}^* N'^!$ in 0 steps, the announced property holds.

Assume that $M = \langle N_1, N_2 \rangle$ with $\sigma = \varphi_1 \otimes \varphi_2$, $a = (a_1, a_2)$, $\Phi \vdash N_i : a_i : \varphi_i$ for $i = 1, 2$. By inductive hypothesis we have $N'_i \in |a_i|^{\varphi_i}$ and hence there are $V_i \in |a_i|_{\mathsf{v}}^{\varphi_i}$ with $N'_i \to_{\mathsf{w}}^* V_i$ for $i = 1, 2$. It follows that $M \to_{\mathsf{w}}^* \langle V_1, V_2 \rangle \in |(a_1, a_2)|_{\mathsf{v}}^{\varphi_1 \otimes \varphi_2}$.

Assume that $M = \mathsf{in}_i N$ with $\sigma = \varphi_1 \oplus \varphi_2$, $a = (i, b)$ and $\Phi \vdash N : b : \varphi_i$. By inductive hypothesis, there exists $V \in |b|_{\mathsf{v}}^{\varphi_i}$ such that $N' \to_{\mathsf{w}}^* V$. We have $\mathsf{in}_i V \in |(i, b)|_{\mathsf{v}}^{\varphi_1 \oplus \varphi_2}$ and $M' = \mathsf{in}_i N' \to_{\mathsf{w}}^* \mathsf{in}_i V$ so that $M' \in |(i, b)|^{\varphi_1 \oplus \varphi_2}$.

Assume that $M = \mathsf{pr}_i N$ with $\sigma = \varphi_i$, $\Phi \vdash N : (a_1, a_2) : \varphi_1 \otimes \varphi_2$ and $a = a_i$. By inductive hypothesis we have $N' \in |(a_1, a_2)|^{\varphi_1 \otimes \varphi_2}$ and hence there are $V_i \in |a_i|_{\mathsf{v}}^{\varphi_i}$ for $i = 1, 2$ such that $N' \to_{\mathsf{w}}^* \langle V_1, V_2 \rangle$. It follows that $M' = \mathsf{pr}_i N' \to_{\mathsf{w}}^* \mathsf{pr}_i \langle V_1, V_2 \rangle \to_{\mathsf{w}} V_i \in |a_i|_{\mathsf{v}}^{\varphi_i}$ and hence $M' \in |a_i|^{\varphi_i}$ as required.

Assume that $M = \mathsf{case}(N, (x_1)N_1, (x_2)N_2)$ with $\Phi \vdash N : (1, b) : \varphi_1 \oplus \varphi_2$ and $\Phi, x_1 : b : \varphi_1 \vdash N_1 : a : \sigma$ (and also $\underline{\Phi}, x_2 : \varphi_2 \vdash N_2 : \sigma$). By inductive hypothesis we have $N' \in |(1, b)|^{\varphi_1 \oplus \varphi_2}$. This means that there is $V \in |b|_{\mathsf{v}}^{\varphi_1}$ such that $N' \to_{\mathsf{w}}^* \mathsf{in}_1 V$. Therefore we have $M' = \mathsf{case}(N', (x_1)N'_1, (x_2)N'_2) \to_{\mathsf{w}}^* \mathsf{case}(\mathsf{in}_1 V, (x_1)N'_1, (x_2)N'_2) \to_{\mathsf{w}} N'_1\,[V/x_1]$. By inductive hypothesis applied to N_1, and because $V \in |b|_{\mathsf{v}}^{\varphi_1}$, we have $N'_1\,[V/x_1] \in |a|^{\sigma}$ and hence $M' \in |a|^{\sigma}$ as expected.

Assume that $M = \langle N \rangle R$ with $\Phi \vdash N : (b, a) : \varphi \multimap \sigma$ and $\Phi \vdash R : b : \varphi$. By inductive hypothesis we have $N' \in |(b, a)|^{\varphi \multimap \sigma}$ and $R' \in |b|^{\varphi}$. Therefore there is $V \in |b|_{\mathsf{v}}^{\varphi}$ such that $R' \to_{\mathsf{w}}^* V$.

Hence $M' = \langle N'\rangle R' \to_{\mathsf{w}}^{*} \langle N'\rangle V \in |a|^{\sigma}$ by definition of $|(b,a)|^{\varphi\multimap\sigma}$ and hence $M' \in |a|^{\sigma}$ by Lemma 14.

Assume that $M = \lambda x^{\varphi} N$ with $\sigma = \varphi \multimap \tau$, $a = (b,c)$ and $\Phi, x : b : \varphi \vdash N : c : \tau$. We must prove that $\lambda x^{\varphi} N' \in |(b,c)|^{\varphi\multimap\tau}$. So let $V \in |b|_{\mathsf{v}}^{\varphi}$, we must check that $\langle \lambda x^{\varphi} N'\rangle V \in |c|^{\tau}$ which results from the fact that $\langle \lambda x^{\varphi} N'\rangle V \to_{\mathsf{w}} N'[V/x] \in |c|^{\tau}$ by inductive hypothesis and from Lemma 14.

Assume last that $M = \mathsf{fix}\, x^{!\sigma} N$ with $\Phi, x : u : !\sigma \vdash N : a : \sigma$ and $\forall b \in u\ \Phi \vdash \mathsf{fix}\, x^{!\sigma} N : b : \sigma$. By inductive hypothesis we have $\mathsf{fix}\, x^{!\sigma} N' \in |b|^{\sigma}$ for each $b \in u$ and therefore $V = (\mathsf{fix}\, x^{!\sigma} N')^{!} \in |u|_{\mathsf{v}}^{!\sigma}$. By inductive hypothesis again we have $N'[V/x] \in |a|^{\sigma}$. Since $\mathsf{fix}\, x^{!\sigma} N' \to_{\mathsf{w}} N'[V/x]$ we get $\mathsf{fix}\, x^{!\sigma} N' \in |a|^{\sigma}$ by Lemma 14 as required. □

So if $\vdash M : \varphi$ and $[M] \neq \emptyset$ we have $M \to_{\mathsf{w}}^{*} V$ for a value V. Let us say that two closed terms M_1, M_2 such that $\vdash M_i : \sigma$ for $i = 1, 2$ are observationally equivalent if for all closed term C of type $!\sigma \multimap 1$, $\langle C\rangle M_1^{!} \to_{\mathsf{w}}^{*} *$ iff $\langle C\rangle M_2^{!} \to_{\mathsf{w}}^{*} *$. As usual, Theorem 16 allows to prove that if $[M_1] = [M_2]$ then M_1 and M_2 are observationally equivalent. It is not hard to prove the converse implication for an extension of CbPV with a non-deterministic superposition operator interpreted as $\cup$ in the Scott Model.

4. A fully polarized version of CBPV

In CbPV positive types are interpreted as !-coalgebras, and general types are simply interpreted as objects of the underlying linear category: in some sense, this system is half-polarized and is intuionistic for that reason. In a fully polarized system we would expect non-positive types to be negative, that is, linear duals of !-coalgebras. Such a system would feature syntactic constructions related to classical logic such as call/cc, the price to pay being a slightly more complicated encoding of data-types.

It is quite easy to turn our hierarchy of types (1) and (2) into a polarized hierarchy:

$$\varphi, \psi, \ldots := !\sigma \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \mathsf{Fix}\, \zeta \cdot \varphi \quad \text{(positive)} \quad (4)$$

$$\sigma, \tau \ldots := ?\varphi \mid \varphi \multimap \sigma \mid \top \quad \text{(negative)} \quad (5)$$

Accordingly we introduce a polarized syntax for expressions featuring five mutually recursive syntactic categories.

$$P, Q \ldots := x \mid N^{!} \mid \langle P_1, P_2\rangle \mid \mathsf{in}_i P \quad \text{(positive terms)}$$

$$M, N, \ldots := \mathsf{der}\, P \mid \lambda x^{\varphi} M \mid \mu\alpha^{\sigma}\, c \mid \mathsf{fix}\, x^{!\sigma} M \quad \text{(negative terms)}$$

$$\pi, \rho \ldots := \alpha \mid \eta^{!} \mid P \cdot \pi \quad \text{(positive contexts)}$$

$$\eta, \theta \ldots := \mathsf{der}\, \pi \mid \mathsf{pr}_i\, \eta \mid [\eta_1, \eta_2] \mid \tilde{\mu} x^{\varphi}\, c \quad \text{(negative contexts)}$$

$$c, d \ldots := P * \eta \mid M * \pi \quad \text{(commands, cuts)}$$

Intuitively, positive terms correspond to data, negative terms to programs, negative contexts to patterns (apart for the negative context $\tilde{\mu} x^{\varphi}\, c$ which generalizes the concept of "closure") and positive contexts to evaluation environments.

The typing rules correspond to a large fragment of LLP, see [17][5], and are given in Figure 6.

Let us say that an expression e is well typed in typing contexts $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_n : \varphi_n)$, $\mathcal{N} = (\alpha_1 : \sigma_1, \ldots, \alpha_k : \sigma_k)$ if e is a positive term P and $\mathcal{P} \vdash P : \varphi \mid \mathcal{N}$ for some type φ, if e is a negative term M and $\mathcal{P} \vdash M : \sigma \mid \mathcal{N}$ for some type σ, if e is a positive context π and $\mathcal{P} \mid \pi : \sigma \vdash \mathcal{N}$ for some type σ, if e is a negative context η and $\mathcal{P} \mid \eta : \varphi \vdash \mathcal{N}$ for some type φ and if e is a command c and $\mathcal{P} \vdash c \mid \mathcal{N}$. In the four first cases, φ (resp. σ) is *the type of* e. Observe that, when it exists, this type is completely determined by e, $\mathcal{P}$ and $\mathcal{N}$ (the typing rules are syntax-directed).

4.1 Operational semantics

The weak reduction rules are given in Figure 7. All redexes are commands and it is crucial to observe that there are no critical pairs. Specifically, there is no command which is simultaneously of shape $M * \pi$ and $P * \tilde{\mu} x^{\varphi}\, c$ because in the former the term is negative whereas it is positive in the latter. In particular the "Lafont critcal pair" $\mu\alpha^{\theta}\, c * \tilde{\mu} x^{\theta}\, d$ cannot occur (θ should be positive and negative!).

Remark: In this weak reduction paradigm, we only reduce commands. A sequence of reduction alternates therefore sequences of *positive commands* of shape $P * \eta$ where a piece of data P is explored by a pattern η with sequences of *negative commands* $M * \pi$ where a program M is executed in an evaluation context π. The transition from the execution phase to the pattern-matching phase is realized by the reduction rule (6) and the converse by (10). We retrieve the basic idea of *focalization* of [1] and of Ludics, [14], that "positive" means passive and "negative", active (many other authors should be mentioned here of course).

We also consider a general reduction relation $\to$ on expressions which is defined by allowing the application of the rules of Figure 7 *anywhere* in an expression as well as the two following $\mu\eta$ reduction rules: $\mu\alpha^{\sigma}\, (M * \alpha) \to M$ if α does not occur free in M and $\tilde{\mu} x^{\varphi}\, (x * \eta) \to \eta$ if x does not occur free in η.

Proposition 17 *If* e *is typable in contexts* $\mathcal{P}$, $\mathcal{N}$ *and* $\mathrm{e} \to \mathrm{e}'$ *then* e' *is typable in contexts* $\mathcal{P}$, $\mathcal{N}$*, belongs to the same syntactic category as* e *and has the same type as* e *(when it applies).*

The proof is a straightforward verification. As usual one has first to state and prove a Substitution Lemma.

Theorem 18 *The reduction relation* $\to$ *on* μCbPV *enjoys the Church-Rosser property.*

The proof uses the usual Tait Martin-Lf method of *parallel reductions* and will be provided in a longer version of this paper. The denotational semantics that we outline now gives us another proof that this calculus is sound.

4.2 Denotational semantics

Assume to be given a model of LL $\mathcal{L}$ as specified in Section 2. With any positive type φ, negative type σ and sequence of pairwise distinct type variables $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)$ containing all free variables of φ and σ, we associate the continuous functors $[\varphi]_{\vec{\zeta}}, [\![\sigma]\!]_{\vec{\zeta}} : (\mathcal{L}^{!}_{\subseteq})^n \to \mathcal{L}^{!}_{\subseteq}$ defined in Figure 8 on objects, the definition on morphisms being similar[6]. With any positive terms and contexts P and π with $\mathcal{P} \vdash P : \varphi \mid \mathcal{N}$ and $\mathcal{P} \mid \pi : \sigma \vdash \mathcal{N}$ we associate coalgebra morphisms $[P]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}^{!}([\mathcal{P}] \otimes [\![\mathcal{N}]\!], [\varphi])$ and $[\pi]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}^{!}([\mathcal{P}] \otimes [\![\mathcal{N}]\!], [\![\sigma]\!])$ and with any negative terms and contexts M and η with $\mathcal{P} \vdash M : \sigma \mid \mathcal{N}$ and $\mathcal{P} \mid \eta : \varphi \vdash \mathcal{N}$ we associate morphisms $[M]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]}, \underline{[\![\sigma]\!]}^{\perp})$ and $[\eta]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]}, \underline{[\varphi]}^{\perp})$. Last, with any command c such that $\mathcal{P} \vdash c \mid \mathcal{N}$ we associate a morphism $[c]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]}, \perp)$.

The interpretation of positive terms is defined as for CbPV (see 2.3). Negative terms: the interpretation of $\mathsf{der}\, P$ uses the dereliction morphism in $\mathcal{L}(\underline{[\varphi]}, ?\underline{[\varphi]})$, the interpretation of $\lambda x^{\varphi} M$ and

[5] In a Sequent Calculus presentation, with double-sided sequents contrarily to most presentations of LLP in the literature. Our syntax is based on the $\lambda\mu\tilde{\mu}$ presentation of sequent calculus-oriented classical λ-calculi due of [4].

[6] Notice that the interpretation of a negative type is actually the semantics of its linear negation; we adopt this convention in order to avoid the explicit introduction of negative objects in the model.

$$\frac{}{\mathcal{P}, x : \varphi \vdash x : \varphi \,|\, \mathcal{N}} \qquad \frac{\mathcal{P} \vdash N : \sigma \,|\, \mathcal{N}}{\mathcal{P} \vdash N^! : !\sigma \,|\, \mathcal{N}} \qquad \frac{\mathcal{P} \vdash P_1 : \varphi_1 \,|\, \mathcal{N} \qquad \mathcal{P} \vdash P_1 : \varphi_2 \,|\, \mathcal{N}}{\mathcal{P} \vdash \langle P_1, P_2 \rangle : \varphi_1 \otimes \varphi_2 \,|\, \mathcal{N}} \qquad \frac{\mathcal{P} \vdash P_i : \varphi_i \,|\, \mathcal{N}}{\mathcal{P} \vdash \mathsf{in}_i P : \varphi_1 \oplus \varphi_2 \,|\, \mathcal{N}}$$

$$\frac{\mathcal{P} \vdash P : \varphi \,|\, \mathcal{N}}{\mathcal{P} \vdash \mathsf{der}\, P : ?\varphi \,|\, \mathcal{N}} \qquad \frac{\mathcal{P}, x : \varphi \vdash M : \sigma \,|\, \mathcal{N}}{\mathcal{P} \vdash \lambda x^{\varphi}\, M : \varphi \multimap \sigma \,|\, \mathcal{N}} \qquad \frac{\mathcal{P} \vdash c \,|\, \alpha : \sigma, \mathcal{N}}{\mathcal{P} \vdash \mu\alpha^{\sigma}\, c : \sigma \,|\, \mathcal{N}} \qquad \frac{\mathcal{P}, x : !\sigma \vdash M : \sigma \,|\, \mathcal{N}}{\mathcal{P} \vdash \mathsf{fix}\, x^{!\sigma}\, M : \sigma \,|\, \mathcal{N}}$$

$$\frac{}{\mathcal{P} \,|\, \alpha : \sigma \vdash \alpha : \sigma, \mathcal{N}} \qquad \frac{\mathcal{P} \,|\, \eta : \varphi \vdash \mathcal{N}}{\mathcal{P} \,|\, \eta^! : ?\varphi \vdash \mathcal{N}} \qquad \frac{\mathcal{P} \vdash P : \varphi \,|\, \mathcal{N} \qquad \mathcal{P} \,|\, \pi : \sigma \vdash \mathcal{N}}{\mathcal{P} \,|\, P \cdot \pi : \varphi \multimap \sigma \vdash \mathcal{N}}$$

$$\frac{\mathcal{P} \,|\, \pi : \sigma \vdash \mathcal{N}}{\mathcal{P} \,|\, \mathsf{der}\, \pi : !\sigma \vdash \mathcal{N}} \qquad \frac{\mathcal{P} \,|\, \eta : \varphi_i \vdash \mathcal{N}}{\mathcal{P} \,|\, \mathsf{pr}_i \eta : \varphi_1 \otimes \varphi_2 \vdash \mathcal{N}} \qquad \frac{\mathcal{P} \,|\, \eta_1 : \varphi_1 \vdash \mathcal{N} \qquad \mathcal{P} \,|\, \eta_2 : \varphi_2 \vdash \mathcal{N}}{\mathcal{P} \,|\, [\eta_1, \eta_2] : \varphi_1 \oplus \varphi_2 \vdash \mathcal{N}} \qquad \frac{\mathcal{P}, x : \varphi \vdash c \,|\, \mathcal{N}}{\mathcal{P} \,|\, \tilde{\mu} x^{\varphi}\, c : \varphi \vdash \mathcal{N}}$$

$$\frac{\mathcal{P} \vdash P : \varphi \,|\, \mathcal{N} \qquad \mathcal{P} \,|\, \eta : \varphi \vdash \mathcal{N}}{\mathcal{P} \vdash P * \eta \,|\, \mathcal{N}} \qquad \frac{\mathcal{P} \vdash M : \sigma \,|\, \mathcal{N} \qquad \mathcal{P} \,|\, \pi : \sigma \vdash \mathcal{N}}{\mathcal{P} \vdash M * \pi \,|\, \mathcal{N}}$$

Figure 6. Typing rules for μCbPV: positive terms, negative terms, positive contexts, negative contexts and commands

$$\mathsf{der}\, P * \eta^! \to_{\mathsf{w}} P * \eta \tag{6}$$
$$\lambda x^{\varphi}\, M * P \cdot \pi \to_{\mathsf{w}} M\,[P/x] * \pi \tag{7}$$
$$\mu\alpha^{\sigma}\, c * \pi \to_{\mathsf{w}} c\,[\pi/\alpha] \tag{8}$$
$$\mathsf{fix}\, x^{!\sigma}\, M * \pi \to_{\mathsf{w}} M\left[\left(\mathsf{fix}\, x^{!\sigma}\, M\right)^! / x\right] * \pi \tag{9}$$
$$M^! * \mathsf{der}\, \pi \to_{\mathsf{w}} M * \pi \tag{10}$$
$$\langle P_1, P_2 \rangle * \mathsf{pr}_i \eta \to_{\mathsf{w}} P_i * \eta \tag{11}$$
$$\mathsf{in}_i P * [\eta_1, \eta_2] \to_{\mathsf{w}} P * \eta_i \tag{12}$$
$$P * \tilde{\mu} x^{\varphi}\, c \to_{\mathsf{w}} c\,[P/x] \tag{13}$$

Figure 7. Reduction rules for μCbPV

$$[\zeta_i]_{\vec{\zeta}}(\vec{P}) = P_i \qquad [!\sigma]_{\vec{\zeta}}(\vec{P}) = !(\underline{[\![\sigma]\!]_{\vec{\zeta}}(\vec{P})^{\perp}})$$
$$[\varphi \otimes \psi]_{\vec{\zeta}}(\vec{P}) = [\varphi]_{\vec{\zeta}}(\vec{P}) \otimes [\psi]_{\vec{\zeta}}(\vec{P})$$
$$[\varphi \oplus \psi]_{\vec{\zeta}}(\vec{P}) = [\varphi]_{\vec{\zeta}}(\vec{P}) \oplus [\psi]_{\vec{\zeta}}(\vec{P})$$
$$[\mathsf{Fix}\, \zeta \cdot \varphi]_{\vec{\zeta}} = \mathrm{Fix}([\varphi]_{\vec{\zeta},\zeta})$$
$$[\![?\varphi]\!]_{\vec{\zeta}}(\vec{P}) = !(\underline{[\varphi]_{\vec{\zeta}}(\vec{P})}^{\perp}) \qquad [\![\varphi \multimap \sigma]\!]_{\vec{\zeta}}(\vec{P}) = [\varphi]_{\vec{\zeta}}(\vec{P}) \otimes [\![\sigma]\!]_{\vec{\zeta}}(\vec{P})$$

Figure 8. Semantics of types in μCbPV

of $\mathsf{fix}\, x^{!\sigma}\, M$ is defined as in CbPV (replacing $[\sigma]$ with $\underline{[\![\sigma]\!]}^{\perp}$) and we provide the interpretation of $\mu\alpha^{\sigma}\, c$: assume that $\mathcal{P} \vdash c \,|\, \mathcal{N}, \alpha : \sigma$ so that $[c]_{\mathcal{P},\mathcal{N},\alpha:\sigma} \in \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]} \otimes \underline{[\![\sigma]\!]}, \perp)$ and we set $[\mu\alpha^{\sigma}\, c]_{\mathcal{P},\mathcal{N}} = \mathrm{cur}\, [c]_{\mathcal{P},\mathcal{N},\alpha:\sigma}$. Positive contexts: we deal only with one case. Assume that $\mathcal{P} \vdash P : \varphi \,|\, \mathcal{N}$ and $\mathcal{P} \,|\, \pi : \sigma \vdash \mathcal{N}$. We have $[P]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}^!([\mathcal{P}] \otimes [\![\mathcal{N}]\!], [\varphi])$ and $[\pi]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}^!([\mathcal{P}] \otimes [\![\mathcal{N}]\!], [\![\sigma]\!])$ so that we can set $[P \cdot \pi]_{\mathcal{P},\mathcal{N}} = ([P]_{\mathcal{P},\mathcal{N}} \otimes [\pi]_{\mathcal{P},\mathcal{N}})\ \mathrm{c}_{[\mathcal{P}] \otimes [\![\mathcal{N}]\!]}$ whose codomain is $[\varphi] \otimes [\![\sigma]\!] = [\![\varphi \multimap \sigma]\!]$ as required. The interpretation of α uses a projection and the interpretation of $\eta^!$ uses a promotion. Negative contexts: assume that $\mathcal{P} \,|\, \pi : \sigma \vdash \mathcal{N}$, then $[\pi]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}^!([\mathcal{P}] \otimes [\![\mathcal{N}]\!], [\![\sigma]\!])$ so we set $[\mathsf{der}\, \pi]_{\mathcal{P},\mathcal{N}} = \mathrm{der}'_{\underline{[\![\sigma]\!]}}\, [\pi]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]}, ?\underline{[\![\sigma]\!]})$ which makes sense since $?\underline{[\![\sigma]\!]} = \underline{[!\sigma]}^{\perp}$ (see Figure 8). The interpretation of $\mathsf{pr}_i\, \eta$ uses $\mathrm{pr}_i^{\perp} \in \mathcal{L}(\underline{[\varphi_i]}^{\perp}, (\underline{[\varphi_1]} \otimes \underline{[\varphi_2]})^{\perp})$. To define $[[\eta_1, \eta_2]]_{\mathcal{P},\mathcal{N}}$ we simply use the pairing operation associated with the cartesian product $\&$ of $\mathcal{L}$ (warning: *not* of $\mathcal{L}^!$!) which is the linear "De Morgan" dual of the coproduct $\oplus$ of $\mathcal{L}$. The interpretation of $\tilde{\mu} x^{\varphi}\, c$ uses a linear curryfication. Commands: assume that $\mathcal{P} \vdash P : \varphi \,|\, \mathcal{N}$ and $\mathcal{P} \,|\, \eta : \varphi \vdash \mathcal{N}$ so that $[P]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}^!([\mathcal{P}] \otimes [\![\mathcal{N}]\!], [\varphi]) \subseteq \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]}, \underline{[\varphi]})$ and $[\eta]_{\mathcal{P},\mathcal{N}} \in \mathcal{L}(\underline{[\mathcal{P}]} \otimes \underline{[\![\mathcal{N}]\!]}, \underline{[\varphi]}^{\perp})$ and we set $[P * \eta]_{\mathcal{P},\mathcal{N}} = \overline{\mathrm{ev}}\, ([\eta]_{\mathcal{P},\mathcal{N}} \otimes [P]_{\mathcal{P},\mathcal{N}})\ \mathrm{c}_{[\mathcal{P}] \otimes [\![\mathcal{N}]\!]}$. The interpretation of $M * \pi$ is similar.

Proposition 19 *Assume that $\mathcal{P} \vdash P : \varphi \,|\, \mathcal{N}$ and that e is a well-typed expression in typing contexts $x : \varphi, \mathcal{P}, \mathcal{N}$. Then we have $[e\,[P/x]]_{\mathcal{P},\mathcal{N}} = [e]_{x:\varphi,\mathcal{P},\mathcal{N}}\, ([P]_{\mathcal{P},\mathcal{N}} \otimes \mathrm{Id})\ \mathrm{c}_{[\mathcal{P}] \otimes [\![\mathcal{N}]\!]}$. Assume that $\mathcal{P} \,|\, \pi : \sigma \vdash \mathcal{N}$ and that e is a well-typed expression in typing contexts $\mathcal{P}$, $\mathcal{N}, \alpha : \sigma$. Then we have $[e\,[\pi/\alpha]]_{\mathcal{P},\mathcal{N}} = [e]_{\mathcal{P},\mathcal{N},\alpha:\sigma}\, (\mathrm{Id} \otimes [\pi]_{\mathcal{P},\mathcal{N}})\ \mathrm{c}_{[\mathcal{P}] \otimes [\![\mathcal{N}]\!]}$.*

The proof of this Substitution Lemma is a simple induction on e, using crucially the fact that $[P]_{\mathcal{P},\mathcal{N}}$ and $[\pi]_{\mathcal{P},\mathcal{N}}$ are morphisms in $\mathcal{L}^!$ and are therefore duplicable and discardable.

Theorem 20 *If e is a well-typed expression in typing contexts $\mathcal{P}$, $\mathcal{N}$ and if $e \to e'$ then $[e]_{\mathcal{P},\mathcal{N}} = [e']_{\mathcal{P},\mathcal{N}}$.*

The proof is routine, using Proposition 19: one checks first the property for each of the 10 redexes (the 8 redexes of Figure 7 and the two $\mu\eta$-redexes of Section 4.1) and then one uses the fact that the interpretation of expressions is defined by structural induction.

A consequence of this easy theorem is that μCbPV is sound in the sense that the reflexive and transitive closure of $\to$ does not equate *eg.* the two booleans $\mathsf{in}_1(\mathsf{fix}\, x^{!\top}\, x^!)$ and $\mathsf{in}_2(\mathsf{fix}\, x^{!\top}\, x^!)$ (closed positive terms of type $!\top \oplus !\top$). Indeed it is easy to build models where these constants have distinct interpretations (for instance the Scott model $\mathbf{Polr}$ of Section 3).

4.3 Translating CbPV into μCbPV

With any positive type φ of CbPV we associate a positive type φ^+ of μCbPV and with any general type σ we associate a negative type of μCbPV. The translation does almost nothing apart adding a few "?" to make sure that σ^- is negative, see Figure 9. Given a CbPV typing context $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_n : \varphi_n)$, we set $\mathcal{P}^+ = (x_1 : \varphi_1^+, \ldots, x_n : \varphi_n^+)$. With any term M of CbPV such that $\mathcal{P} \vdash M : \sigma$ we explain now how to associate a term M^- of μCbPV such that $\mathcal{P}^+ \vdash M^- : \sigma^- \,|\,$.

If M is a variable x typed by $\mathcal{P}, x : \sigma \vdash x : \sigma$, we set

$$x^- = \mathsf{der}\, x\,.$$

$$\zeta^+ = \zeta \qquad (!\sigma)^+ = !(\sigma^-)$$
$$(\varphi_1 \otimes \varphi_2)^+ = \varphi_1^+ \otimes \varphi_2^+ \qquad (\varphi_1 \oplus \varphi_2)^+ = \varphi_1^+ \oplus \varphi_2^+$$
$$(\mathsf{Fix}\,\zeta \cdot \varphi)^+ = \mathsf{Fix}\,\zeta \cdot \varphi^+$$
$$\varphi^- = ?(\varphi^+) \qquad (\varphi \multimap \sigma)^- = \varphi^+ \multimap \sigma^-$$

Figure 9. Translation of types

If $M = N^!$ with $\sigma = !\tau$ and $\mathcal{P} \vdash N : \tau$ then by inductive hypothesis $\mathcal{P}^+ \vdash N^- : \tau^- \,|$. Therefore $\mathcal{P}^+ \vdash (N^-)^! : !(\tau^-)\,|$ and we set
$$(N^!)^- = (N^-)^! \,.$$

If $M = \mathsf{der}(N)$ and $\mathcal{P} \vdash N : !\sigma$ then by inductive hypothesis $\mathcal{P}^+ \vdash N^- : ?!\sigma^- \,|$. We have $\mathcal{P}^+ \,|\, \alpha : \sigma^- \vdash \alpha : \sigma^-$ and hence $\mathcal{P}^+ \,|\, (\mathsf{der}\,\alpha)^! : ?!\sigma^- \vdash \alpha : \sigma^-$, and we set
$$\mathsf{der}(N)^- = \mu\alpha^{\sigma^-}\,(N^- * (\mathsf{der}\,\alpha)^!) \,.$$

If $M = \lambda x^\varphi\, N$ with $\sigma = \varphi \multimap \tau$ and $\mathcal{P}, x : \varphi \vdash N : \sigma$ then inductive hypothesis $\mathcal{P}^+, x : \varphi^+ \vdash N^- : \sigma^- \,|$ so that $\mathcal{P}^+ \vdash \lambda x^{\varphi^+}\, N^- : \varphi^+ \multimap \sigma^- \,|$ and we set $(\lambda x^\varphi\, N)^- = \lambda x^{\varphi^+}\, N^-$.

If $M = \langle N \rangle R$ with $\mathcal{P} \vdash N : \varphi \multimap \sigma$ and $\mathcal{P} \vdash R : \varphi$ then by inductive hypothesis, $\mathcal{P}^+ \vdash N^- : \varphi^+ \multimap \sigma^- \,|$ and $\mathcal{P}^+ \vdash R^- : ?\varphi^+ \,|$. We have $\mathcal{P}^+, x : \varphi^+ \,|\, x \cdot \alpha : \varphi^+ \multimap \sigma^- \vdash \alpha : \sigma^-$, hence $\mathcal{P}^+, x : \varphi^+ \vdash N^- * (x \cdot \alpha) \,|\, \alpha : \sigma^-$. Therefore we have $\mathcal{P}^+ \,|\, \tilde{\mu} x^{\varphi^+}\, N^- * (x \cdot \alpha) : \varphi^+ \vdash \alpha : \sigma^-$ so that $\mathcal{P}^+ \,|\, (\tilde{\mu} x^{\varphi^+}\, N^- * (x \cdot \alpha))^! : ?\varphi^+ \vdash \alpha : \sigma^-$ and we set
$$\langle N \rangle R^- = \mu\alpha^{\sigma^-}\,(R^- * (\tilde{\mu} x^{\varphi^+}\, N^- * (x \cdot \alpha))^!) \,.$$

If $M = \mathsf{in}_i N$ with $\mathcal{P} \vdash N : \varphi_i$ then by inductive hypothesis $\mathcal{P}^+ \vdash N^- : ?\varphi_i^+ \,|$. One has $\mathcal{P}^+ \,|\, \tilde{\mu} x^{\varphi_i^+}\,(\mathsf{in}_i x * \alpha) : \varphi_i^+ \vdash \alpha : \varphi_1^+ \oplus \varphi_2^+$ hence $\mathcal{P}^+ \vdash N^- * (\tilde{\mu} x^{\varphi_i^+}\,(\mathsf{in}_i x * \alpha))^! \,|\, \alpha : \varphi_1^+ \oplus \varphi_2^+$ so
$$(\mathsf{in}_i N)^- = \mathsf{der}\,(\mu\alpha^{\varphi_1^+ \oplus \varphi_2^+}\, N^- * (\tilde{\mu} x^{\varphi_i^+}\,(\mathsf{in}_i x * \alpha))^!) \,.$$

If $M = \mathsf{case}(N, (x_1)M_1, (x_2)M_2)$ with $\mathcal{P} \vdash N : \varphi_1 \oplus \varphi_2$ and $\mathcal{P}, x_i : \varphi_i \vdash M_i : \sigma$ then $\mathcal{P}^+ \vdash N^- : ?(\varphi_1^+ \oplus \varphi_2^+)\,|$ and $\mathcal{P}^+, x_i : \varphi_i^+ \vdash M_i^- : \sigma^- \,|$. We have $\mathcal{P}^+ \,|\, \tilde{\mu} x_i^{\varphi_i^+}\, M_i^- * \alpha : \varphi_i^+ \vdash \alpha : \sigma^-$ and hence $\mathcal{P}^+ \,|\, \left[\tilde{\mu} x_1^{\varphi_1^+}\, M_1^- * \alpha, \tilde{\mu} x_2^{\varphi_2^+}\, M_2^- * \alpha\right] : \varphi_1^+ \oplus \varphi_2^+ \vdash \alpha : \sigma^-$ for $i = 1, 2$, so
$$\begin{aligned}&\mathsf{case}(N, (x_1)M_1, (x_2)M_2)^- \\ &\quad = \mu\alpha^{\sigma^-}\, N^- * \left[\tilde{\mu} x_1^{\varphi_1^+}\, M_1^- * \alpha, \tilde{\mu} x_2^{\varphi_2^+}\, M_2^- * \alpha\right]^! .\end{aligned}$$

If $M = \langle M_1, M_2 \rangle$ there are two possible translations (similar phenomena occur in CPS translation, see for instance [25]), a *left first* translation and a *right first* translation. We give the first one, the other one being obtained by swapping the roles of M_1 and M_2. We assume that $\mathcal{P} \vdash M_i : \varphi_i$ and hence $\mathcal{P}^+ \vdash M_i^- : ?\varphi_i^+ \,|$ for $i = 1, 2$. We have $\mathcal{P}^+, x_2 : \varphi_2^+ \,|\, \tilde{\mu} x_1^{\varphi_1^+}\, \mathsf{der}\,\langle x_1, x_2 \rangle * \alpha : \varphi_1^+ \vdash \alpha : ?(\varphi_1^+ \otimes \varphi_2^+)$, hence $\mathcal{P}^+, x_2 : \varphi_2^+ \vdash M_1^- * (\tilde{\mu} x_1^{\varphi_1^+}\, \mathsf{der}\,\langle x_1, x_2 \rangle * \alpha)^! \,|\, \alpha : ?(\varphi_1^+ \otimes \varphi_2^+)$ and hence we set
$$\begin{aligned}&\langle M_1, M_2 \rangle^- = \mu\alpha^{?(\varphi_1^+ \otimes \varphi_2^+)} \\ &\qquad M_2^- * \left(\tilde{\mu} x_2^{\varphi_2^+}\, M_1^- * (\tilde{\mu} x_1^{\varphi_1^+}\, \mathsf{der}\,\langle x_1, x_2 \rangle * \alpha)^!\right)^! .\end{aligned}$$

If $M = \mathsf{pr}_i N$ where $\mathcal{P} \vdash N : \varphi_1 \otimes \varphi_2$ then $\mathcal{P}^+ \vdash N^- : ?(\varphi_1^+ \otimes \varphi_2^+)\,|$. We have $\mathcal{P}^+ \,|\, \tilde{\mu} x^{\varphi_i^+}\,(\mathsf{der}\,x * \alpha) : \varphi_i^+ \vdash \alpha : ?\varphi_i^+$ so that $\mathcal{P}^+ \,|\, (\mathsf{pr}_i\, \tilde{\mu} x^{\varphi_i^+}\,(\mathsf{der}\,x * \alpha))^! : ?(\varphi_1^+ \otimes \varphi_2^+) \vdash \alpha : ?\varphi_i^+$ so
$$(\mathsf{pr}_i N)^- = \mu\alpha^{?\varphi_i^+}\, N^- * (\mathsf{pr}_i\, \tilde{\mu} x^{\varphi_i^+}\,(\mathsf{der}\,x * \alpha))^! \,.$$

If $M = \mathsf{fix}\,x^{!\sigma}\, N$ with $\mathcal{P}, x : !\sigma \vdash N : \sigma$ then $\mathcal{P}^+, x : !(\sigma^-) \vdash N^- : \sigma^- \,|$, so $(\mathsf{fix}\,x^{!\sigma}\, N)^- = \mathsf{fix}\,x^{!(\sigma^-)}\, N^-$.

Given a CbPV value V such that $\mathcal{P} \vdash V : \varphi$, one can straightforwardly define a positive $\mu\mathsf{CbPV}$ term V^+ such that $\mathcal{P}^+ \vdash V^+ : \varphi^+ \,|$ as follows: $x^+ = x$, $(M^!)^+ = (M^-)^!$, $\langle V_1, V_2 \rangle^+ = \langle V_1^+, V_2^+ \rangle$ and $(\mathsf{in}_i V)^+ = \mathsf{in}_i(V^+)$ for $i = 1, 2$.

Lemma 21 *If $\mathcal{P} \vdash V : \varphi$ in CbPV and if V is a value then $V^- \to^* \mathsf{der}\,(V^+)$.*

Proof. Simple verification. For instance, if $V = \langle V_1, V_2 \rangle$, then
$$\begin{aligned}V^- &\to^* \mu\alpha\, \mathsf{der}\,(V_2^+) * (\tilde{\mu} x_2\, \mathsf{der}\, V_1^+ * (\tilde{\mu} x_1\, \mathsf{der}\,\langle x_1, x_2 \rangle * \alpha)^!)^! \\ &\to \mu\alpha\, V_2^+ * \tilde{\mu} x_2\,(\mathsf{der}\, V_1^+ * (\tilde{\mu} x_1\, \mathsf{der}\,\langle x_1, x_2 \rangle * \alpha)^!) \\ &\to \mu\alpha\, \mathsf{der}\, V_1^+ * (\tilde{\mu} x_1\, \mathsf{der}\,\langle x_1, V_2^+ \rangle * \alpha)^! \to^* \langle V_1^+, V_2^+ \rangle \,.\end{aligned}$$
□

Lemma 22 *If $\mathcal{P}, x : \varphi \vdash M : \sigma$ and $\mathcal{P} \vdash V : \varphi$ in CbPV (with V being a value) then $M\,[V/x]^- \to^* M^-\,\left[V^+/x\right]$.*

Proof. Induction on M, using Lemma 21 when $M = x$. □

The translation above respects the operational semantics.

Theorem 23 *If $\mathcal{P} \vdash M : \sigma$ in CbPV then $\mathcal{P}^+ \vdash M^- : \sigma^- \,|$. Moreover, if $M \to_{\mathsf{w}} M'$ then there is a negative term R of $\mu\mathsf{CbPV}$ such that $M^- \to^* R$ and $M'^- \to^* R$. .*

Proof. The first statement is essentially proven in the definition above of the translation. The proof of the second statement is a simple inspection of the reduction rules of CbPV. Let us consider two cases. Assume first that $M = \langle \lambda x\, N \rangle V$, then $M^- = \mu\alpha\, V^- * (\tilde{\mu} x\,(\lambda x\, N^-) * x \cdot \alpha)^! \to^* \mu\alpha\, \mathsf{der}\, V^+ * (\tilde{\mu} x\,(\lambda x\, N^-) * x \cdot \alpha)^!$ by Lemma 21. Hence $M^- \to^* N^-\,\left[V^+/x\right]$. On the other hand we have $M \to_{\mathsf{w}} N\,[V/x]$ in CbPV, and $N\,[V/x]^- \to^* N^-\,\left[V^+/x\right]$ by Lemma 22.

Assume now $M = \mathsf{pr}_i \langle V_1, V_2 \rangle$, we have $M^- = \mu\alpha\, \langle V_1, V_2 \rangle^- * (\mathsf{pr}_i\, \tilde{\mu} x\,(\mathsf{der}\,x * \alpha))^! \to^* \mu\alpha\, \mathsf{der}\,\langle V_1^+, V_2^+ \rangle * (\mathsf{pr}_i\, \tilde{\mu} x\,(\mathsf{der}\,x * \alpha))^! \to^* \mathsf{der}\, V_i^+$ by Lemma 21. On the other hand $M \to_{\mathsf{w}} V_i$ and $V_i^- \to^* \mathsf{der}\, V_i^+$ by Lemma 22. The other cases are similar. □

As a consequence, using Theorem 18 one proves that, if $M \to^* M'$ in CbPV there is a negative term R of $\mu\mathsf{CbPV}$ such that $M^- \to^* R$ and $M'^- \to^* R$ (induction on the length of the reduction $M \to^* M'$). This shows that CbPV embeds in $\mu\mathsf{CbPV}$ by a translation $M \mapsto M^-$ which is compatible with the operational semantics. In the long version of this paper, we will also describe a simple relation between the semantics of M and of M^-.

Conclusion

The half-polarized and fully polarized presentations of CBPV proposed in this work admit LL-based models featuring non-trivial effects such as non-deterministic computations, or probabilistic computations, see [7] with full abstraction properties, see [10]. Accommodating other effects such as global and local states will require a deeper semantical analysis which could be based on the very nice combination of effect monads with linearity developed in the *enriched effect calculus* of [8].

A. Appendix: ommitted proofs

A.1 Proof of Theorem 8

Proof. We check first that (S, p_S) is a !-coalgebra. The relation p_S is a morphism in $\mathbf{Polr}$: if $(a, u) \in \mathrm{p}_S$ and (a', u') satisfy $a \leq_S a'$ and $u' \leq_{!S} u$, then for all $b' \in u'$ there exists $b \in u$ such that $b' \leq_S b$. Then we know that $b \leq_S a$ because $(a, u) \in \mathrm{p}_S$ and finally $a \leq_S a'$ by our assumption. Hence $b' \leq_S a'$ and therefore $(a', u') \in \mathrm{p}_S$ as contended.

Let $(a, a') \in |S| \times |S|$. If $(a, a') \in \mathrm{der}_S\ \mathrm{p}_S$ then there exists $u \in |!S|$ such that $\forall b \in u\ b \leq_S a$, and there is $b \in u$ such that $a' \leq_S b$. Therefore $a' \leq_S a$ and it follows that $\mathrm{der}_S\ \mathrm{p}_S \subseteq \mathrm{Id}_S$. Assume conversely that $a' \leq_S a$, taking $u = \{a\}$ (or $u = \{a'\}$), we see that $(a, a') \in \mathrm{der}_S\ \mathrm{p}_S$, hence we have $\mathrm{der}_S\ \mathrm{p}_S = \mathrm{Id}_S$.

Let now $(a, \mathcal{U}) \in |S| \times |!!S|$ with $\mathcal{U} = \{u_1, \dots, u_n\}$. Assume first that $(a, \mathcal{U}) \in \mathrm{dig}_S\ \mathrm{p}_S$. There is $u \in |!S|$ such that $u_i \leq_{!S} u$ for each $i = 1, \dots, n$, and $b \leq_S a$ for each $b \in u$. It follows that $(a, u_i) \in \mathrm{p}_S$ for each i and hence $(\{a\}, \mathcal{U}) \in !\mathrm{p}_S$. Since $(a, \{a\}) \in \mathrm{p}_S$ it follows that $(a, \mathcal{U}) \in !\mathrm{p}_S\, \mathrm{p}_S$. Assume conversely that $(a, \mathcal{U}) \in !\mathrm{p}_S\, \mathrm{p}_S$. So there is $u \in |!S|$ such that $b \leq_S a$ for all $b \in u$ and, for each i, there exists $b_i \in u$ such that $a' \leq b_i$ for each $a' \in u_i$. So we have $u_i \leq_{!S} u$ for each i and hence $(u, \mathcal{U}) \in \mathrm{dig}_S$ and hence $(a, \mathcal{U}) \in \mathrm{dig}_S\ \mathrm{p}_S$. This ends the proof that (S, p_S) is a !-coalgebra.

We come now to the second statement. Let P be a !-coalgebra and let $S = \underline{P}$. Let $(a, u) \in \mathrm{h}_P$, we prove that $b \leq_S a$ for all $b \in u$. Let $b \in u$, we have $(u, b) \in \mathrm{der}_S$ by definition of this morphism and hence $(a, b) \in \mathrm{der}_S\ \mathrm{h}_P = \mathrm{Id}_S$ and hence $b \leq_S a$. This shows that $\mathrm{h}_P \subseteq \mathrm{p}_S$.

Before proving the converse inclusion, we make a useful observation. Let $b \in |S|$. We have $(b, b) \in \mathrm{Id}_S = \mathrm{der}_S\ \mathrm{h}_P$ and hence there exists u such that $(b, u) \in \mathrm{h}_P$ and $(u, b) \in \mathrm{der}_S$, which implies that there is $b' \in u$ such that $b \leq_S b'$, hence $\{b\} \leq_{!S} u$. Since $(b, u) \in \mathrm{h}_P \in \mathbf{Polr}(S, !S)$, we have $(b, \{b\}) \in \mathrm{h}_P$.

Assume now that $(a, u) \in \mathrm{p}_S$ with $u = \{a_1, \dots, a_n\}$. We have seen that $(a, \{a\}) \in \mathrm{h}_P$ and hence $(a, \{a_i\})$ for $i = 1, \dots, n$ since $a_i \leq_S a$ for each i. Therefore $(\{a\}, \mathcal{U}) \in !\mathrm{h}_P$ where $\mathcal{U} = \{\{a_1\}, \dots, \{a_n\}\}$. Hence $(a, \mathcal{U}) \in !\mathrm{h}_P\, \mathrm{h}_P = \mathrm{dig}_S\ \mathrm{h}_P$. Hence there is $u' \in |!S|$ such that $(a, u') \in \mathrm{h}_P$ and $(u', \mathcal{U}) \in \mathrm{dig}_S$. This latter property means that $u \leq_{!S} u'$ and hence we have $(a, u) \in \mathrm{h}_P$ as contended. ☐

A.2 Proof of Theorem 10

Proof. Let $f \in \mathbf{Polr}^!(S, T)$. This means that $\mathrm{p}_T\, f = !f\, \mathrm{p}_S$. Let $\xi \in \mathsf{Idl}(S)$. We prove that the set $f\, \xi = \{b \mid \exists a \in \xi\ (a, b) \in f\} \subseteq |T|$ is $\leq_T$-directed. Let indeed $b_1, b_2 \in f\, \xi$. Let $a_i \in \xi$ be such that $(a_i, b_i) \in f$ for $i = 1, 2$. Since ξ is directed, there is $a \in \xi$ such that $a_i \leq_S a$ for $i = 1, 2$. Then we have $(a, \{a_1, a_2\}) \in \mathrm{p}_S$, and of course $(\{a_1, a_2\}, \{b_1, b_2\}) \in !f$. Since f is a !-morphism, we must have $(a, \{b_1, b_2\}) \in \mathrm{p}_T\, f$. This means that there is $b \in |T|$ such that $(a, b) \in f$ and $b_i \leq_T b$ for $i = 1, 2$. Hence $f\, \xi$ is directed as contended. The map $\xi \mapsto f\, \xi$ is obviously Scott-continuous from $\mathsf{Idl}(S)$ to $\mathsf{Idl}(T)$; we denote it as $\mathsf{C}(f)$. This operation on morphisms is clearly functorial.

Conversely, let $\varphi : \mathsf{Idl}(S) \to \mathsf{Idl}(T)$ be Scott continuous. We set $\mathsf{A}(\varphi) = \{(a, b) \in |S| \times |T| \mid b \in \varphi(\downarrow a)\}$ and we prove that $\mathsf{A}(f) \in \mathbf{Polr}^!(S, T)$. The first thing to check is that $\mathsf{A}(\varphi) \in \mathbf{Polr}(S, T)$ but this results immediately from the definition and from the fact that a Scott-continuous function is monotone. It remains to prove that $!\mathsf{A}(\varphi)\, \mathrm{p}_S = \mathrm{p}_T\, \mathsf{A}(\varphi)$. Let $a \in |S|$ and $v = \{b_1, \dots, b_n\} \in |!T|$. Assume first that $(a, v) \in !\mathsf{A}(\varphi)\, \mathrm{p}_S$. Let $u \in |!S|$ be such that $(a, u) \in \mathrm{p}_S$ and $(u, v) \in !\mathsf{A}(\varphi)$. Then we have $v \subseteq \bigcup_{a' \in u} \varphi(\downarrow a') \subseteq \varphi(\downarrow a)$ by applying the definitions and using the fact that φ is monotone. But $\varphi(\downarrow a)$ is directed in T and v is finite. Hence there is $b \in |T|$ such that $(a, b) \in \mathsf{A}(\varphi)$ and $(b, v) \in \mathrm{h}_T$ as required. Assume next that $(a, v) \in \mathrm{p}_T\, \mathsf{A}(\varphi)$. This means that there is $b \in |T|$ such that $b \in \varphi(\downarrow a)$ and $(b, v) \in \mathrm{p}_T$. So we have $\forall b' \in v\ (a, b') \in \mathsf{A}(\varphi)$ and hence $(\{a\}, v) \in !\mathsf{A}(\varphi)$. Since $(a, \{a\}) \in \mathrm{p}_S$ we have $(a, v) \in !\mathsf{A}(\varphi)\, \mathrm{p}_S$ as required.

We prove now that these two operations are inverse of each other. Let first $f \in \mathbf{Polr}^!(S, T)$ and let $\varphi = \mathsf{C}(f)$. Let $(a, b) \in \mathsf{A}(\varphi)$. This means that $b \in \varphi(\downarrow a)$, that is $b \in f \downarrow a$ and hence $(a, b) \in f$. Conversely let $(a, b) \in f$. Then we have $b \in f \downarrow a$ and hence $(a, b) \in \mathsf{A}(\varphi)$. We have proven that $\mathsf{A}(\mathsf{C}(f)) = f$. Let $\varphi : \mathsf{Idl}(S) \to \mathsf{Idl}(T)$ be Scott continuous. Let $f = \mathsf{A}(\varphi) \in \mathbf{Polr}^!(S, T)$. Let $\xi \in \mathsf{Idl}(S)$, we have $\mathsf{C}(f)(\xi) = f\, \xi = \cup_{a \in \xi} \varphi(\downarrow a)$. The set $\{\downarrow a \mid a \in \xi\}$ is a directed subset of $\mathsf{Idl}(S)$ whose lub is ξ. Since φ is Scott continuous we have $\mathsf{C}(f)(\xi) = \varphi(\xi)$, and since this is true for all $\xi \in \mathsf{Idl}(S)$ we have $\mathsf{C}(\mathsf{A}(\varphi)) = \varphi$.

Let $f, f' \in \mathbf{Polr}^!(S, T)$ be such that $f \subseteq f'$. Let $\xi \in \mathsf{Idl}(S)$. Let $b \in \mathsf{C}(f)(\xi)$. This means that $b \in f\, \xi$ and hence $b \in f'\, \xi$, so we have $\mathsf{C}(f) \leq \mathsf{C}(f')$ for the pointwise order of functions. Let $\varphi, \varphi' : \mathsf{Idl}(S) \to \mathsf{Idl}(T)$ be such that $\varphi \leq \varphi'$ for that order on functions. Let $(a, b) \in \mathsf{A}(\varphi)$, this means that $b \in \varphi(\downarrow a)$. By our assumption we have $b \in \varphi'(\downarrow a)$ and hence $(a, b) \in \mathsf{A}(\varphi')$. So $\mathsf{A}(\varphi) \subseteq \mathsf{A}(\varphi')$. This ends the proof of the theorem. ☐

A.3 Proof of Lemma 11

Proof. Assume first that $f_1 = f_2\, \varphi^+_{S_1, S_2}$. Let $(a, b) \in f_1$, we have $(a, b) \in f_2\, \varphi^+_{S_1, S_2}$ and so there is $a' \in |S_2|$ such that $(a, a') \in \varphi^+_{S_1, S_2}$ (that is $a' \leq_{S_2} a$) and $(a', b) \in f_2$. Since $f_2 \in \mathbf{Polr}(S_2, T)$ we have $(a, b) \in f_2$ so $f_1 \subseteq f_2 \cap |S_1 \multimap T|$. Let now $(a, b) \in f_2 \cap |S_1 \multimap T|$, we have $(a, a) \in \varphi^+_{S_1, S_2}$ and hence $(a, b) \in f_1$. We prove now the converse implication, so assume that $f_1 = f_2 \cap |S_1 \multimap T|$. Let $(a, b) \in f_1$, we know that $(a, b) \in f_2$ and that $(a, a) \in \varphi^+_{S_1, S_2}$ and hence $(a, b) \in f_2\, \varphi^+_{S_1, S_2}$. Conversely let $(a, b) \in f_2\, \varphi^+_{S_1, S_2}$. Let $a' \in |S_2|$ be such that $(a, a') \in \varphi^+_{S_1, S_2}$ (and hence $a' \leq_{S_2} a$) and $(a', b) \in f_2$. Since $f_2 \in \mathbf{Polr}(S_2, T)$ we have $(a, b) \in f_2$ so $(a, b) \in f_2 \cap |S_1 \multimap T| = f_1$. ☐

Acknowledgments

I would like to thank Antonio Bucciarelli, Pierre-Louis Curien, Giulio Guerrieri, Jean-Louis Krivine, Paul-André Melliès, Michele Pagani and Christine Tasson for deep and enlightening discussions on the ideas developed in this article.

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